Geometry and Foams: 2D Dynamics and 3D Statics

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We discuss the implications of the classical Gauss-Bonnet formula to foam in two and three dimensions. For a two-dimensional foam it gives a generalization of the von Neumann law for the coarsening of foams to curved surfaces. As a consequence of this we find that the stability properties of stationary bubbles of such a froth depend on the Gaussian curvature of the surface. For three-dimensional foam we find a relation between the average Gaussian curvature of a soap film and the average number of vertices for each face.

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Foams are ubiquitous, aesthetically pleasing, and serve as a paradigm for a wide range of physical phenomena [1]. Among the manifold uses of foams in technology are filtration, insulation, and as substrates for chemical reactions. The coarsening of two-dimensional (2D) foams is a model for the growth of crystalline grains in thin metallic films [2]. In this paper we describe two principal results about soap froth. The first and main result, Eq. (3) below, is a generalization of the von Neumann coarsening law [3]. It describes the evolution of *two-dimensional* foam on a curved surface due to the diffusive gas transfer between bubbles. The second, Eq. (5) below, relates $\langle n \rangle$, the average number of vertices per face, and the average Gaussian curvature, $\langle K \rangle$, of the soap films, for *three-dimensional* foams. Equation (3) has interesting consequences for the stability properties of "large" bubbles, which can be stable or unstable depending on the Gaussian curvature of the surface to which they are confined. Equation (5) has the consequence that it gives the bound $\langle n \rangle > 5.104\ldots$ for isobaric foam. Equations (3) and (5) are united by the fact that both ultimately trace back to the classical Gauss-Bonnet formula. It is interesting that in two dimensions the Gauss-Bonnet formula leads to a dynamical law while in three dimensions it leads to a law about statistical (static) properties.

The study of foam coarsening on curved surfaces is of some interest in the context of grain growth of thin coatings of surfaces [2]. The von Neumann law has proven to be a very useful tool in the study of (planar) grain growth, although the reasons for its success are not fully understood. As a consequence, it is not clear if this success is special to the planar case, or whether the soap froth analogy also gives a good description of grain growth on curved surfaces. Thus, tests of Eq. (3) and its ramifications would contribute to a better understanding of the validity of the soap froth model of grain growth.

(1) Coarsening of 2D foams on curved surfaces.—The von Neumann law was originally derived in 1951 [3] for planar foam such as might be obtained by confining soap solution between two planar parallel glass plates [4]. It states that, for each bubble, $dN/dt = (\pi\kappa\sigma/3)(n-6)$. Here N is the number of gas molecules inside the bubble, n is the number of vertices it has, σ is the line tension, and κ is the diffusion constant of the gas through the bubble walls. This law is remarkable in its simplicity, exhibiting no dependence whatsoever on the bubble's neighbors, nor on its own detailed shape or size. Our first result is concerned with the coarsening of foams constrained to curved surfaces, and the generalization of the von Neumann law to such cases.

Consider a two-dimensional foam constrained to lie on a surface Σ, which is smooth, but otherwise arbitrary. The foam (sometimes called a "late stage" or "polygonal" foam) consists of a set of vertices connected by edges, three of which emanate from each vertex. We require that at all stages in its evolution the foam be in mechanical equilibrium; that is, that there be no net force on either vertices or edges. We shall consider idealized foams in the sense that the edges are all identical (in the sense that σ and κ are the same), of zero thickness, with pointlike vertices, and with no Plateau borders. The conditions for the balance of forces imply Plateau's laws, namely, (1) the angle between a pair of edges common to a vertex is $2\pi/3$ (this holds also on curved surfaces); (2) the edges, *on a flat surface*, are sections of circles. As we shall see, on a curved surface, the generalization of (2) is that edges have *constant geodesic curvature*.

Mechanical equilibrium of an edge means that at each point x of the edge the forces $\mathbf{F}_P$ (due to the pressure difference, $\Delta P \equiv P_{\text{in}} - P_{\text{out}}$, across the edge) and $\mathbf{F}_\sigma$ (due to the line tension on the arc) are balanced on the tangent plane at x. For the planar case this yields the Young-Laplace equation $\Delta P = \sigma/R$, where R is the (signed) radius of the circular arc. This relation must be modified for nonflat surfaces.

To facilitate our discussion, we introduce an orthonormal triad of vectors defined at each point x of an edge (see Fig. 1). Let $\mathbf{n}$ be the unit normal to the surface Σ, $\mathbf{t}$ the unit tangent to the edge (oriented so that the edges of the bubble are traversed counterclockwise), and $\boldsymbol{\gamma} \equiv \mathbf{n} \times \mathbf{t}$. The vectors $\mathbf{t}$ and $\boldsymbol{\gamma}$ span the tangent plane at x. Consider an elemental portion of an edge of arc length δl. The force $\mathbf{F}_P$ lies in the tangent plane, and is parallel to $\boldsymbol{\gamma}$; its magnitude is $\Delta P \delta l$. Hence $\mathbf{F}_P = -\Delta P \delta l \boldsymbol{\gamma}$. The force $\mathbf{F}_\sigma$

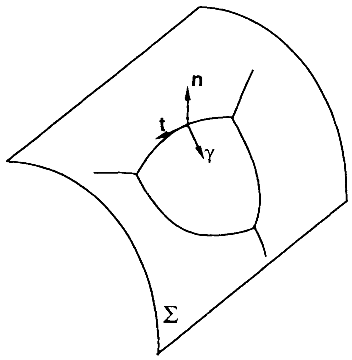

FIG. 1. A curved surface, Σ with the triad of vectors $(\mathbf{n},\mathbf{t},\boldsymbol{\gamma})$.

is the resultant of the two forces acting in the tangential direction at each end of the portion of the arc. At each end, the line tension exerts a force $\sigma\mathbf{t}(x)$, so the total force $\mathbf{F}_\sigma(x)=\sigma(\mathbf{t}(x+\delta l/2)-\mathbf{t}(x-\delta l/2))=\sigma(d\mathbf{t}/dx)\delta l$. If the surface is flat, $d\mathbf{t}/dx$ is parallel to $\boldsymbol{\gamma}$, but in general it has a component in the $\mathbf{n}$ direction, which, since the foam is assumed restricted to the surface, is balanced by forces normal to the substrate. Hence, the balance of forces takes place in the tangent plane, and yields

$$\Delta P=\sigma C_g\,, \tag{1}$$

where $C_g\equiv\boldsymbol{\gamma}\cdot d\mathbf{t}/dx$ is the *geodesic curvature* of the edge at x [5]. Since ΔP is constant along an edge, the edges have constant geodesic curvature. In the planar case we recover the Young-Laplace law, since $C_g=1/R$.

An alternative, thermodynamic derivation of Eq. (1) makes use of the minimization of the free energy

$$\sum_{j\in\text{bubbles}}F(T,A_j,N_j)+\sum_{j\in\text{edges}}\sigma(T)l_j\,,$$

where $F(T,A,N)$ is the (bulk 2D) Helmholtz free energy of a bubble at temperature T, of area A, and containing N gas molecules. l_j is the length of the jth edge. Thermodynamic equilibrium says that the total free energy is stationary under variation of the jth edge. Variation of l_j leads to a variation in A_{out} and A_{in}, the areas of the two bubbles having l_j as an edge, so $\delta F(T,A_{\text{out}},N_{\text{out}})+\delta F(T,A_{\text{in}},N_{\text{in}})+\sigma\delta l_j=0$. Now since $\delta F(T,A_{\text{out}},N_{\text{out}})=-P_{\text{out}}\delta A_{\text{out}}$ (and similarly for the "in" bubble) and $\delta(A_{\text{out}}+A_{\text{in}})=0$ we get $-\Delta P\delta A_{\text{in}}+\sigma\delta l=0$. This yields Eq. (1) since $C_g=\delta l/\delta A$ (see, e.g., [5], Vol. 4, Chap. 9).

We assume that a bubble evolves owing to pressure-difference–induced diffusion of gas between neighboring bubbles. That is, $dN/dt=-\kappa\sum_k\Delta P_k l_k$, where the sum is over the edges of the bubble. We next observe [6] that this is precisely one of the terms in the Gauss-Bonnet theorem applied to a (polygonlike) bubble B on a surface. The Gauss-Bonnet theorem states that

$$\int_B K\,dA+\int_{\partial B}C_g\,dx+\sum_j(\pi-\alpha_j)=2\pi\,, \tag{2}$$

where K is the Gaussian curvature of the surface (i.e., $K\equiv 1/R_1R_2$, where $R_{1,2}$ are the two principal radii of curvatures), B is the polygon's surface, ∂B is its perimeter, and the sum is over the internal angles α_j at its vertices.

Applying Eq. (1) to the relation for diffusive transfer, we obtain $dN/dt=-\kappa\sigma\sum_k C_g(k)l_k$. Combined with Eq. (2) (specialized for the case where C_g is constant over the various edges comprising the bubble perimeter, and all angles $\alpha_j=2\pi/3$), we obtain our main result:

$$\frac{1}{\kappa\sigma}\frac{dN}{dt}=\frac{\pi}{3}(n-6)+\int_B K\,dA\,. \tag{3}$$

When the surface is flat, $K=0$, and we recover the von Neumann law. It should be noted that, for the planar case, this same equation governs grain growth in thin metallic films [7] (and, e.g., the growth of a grain of some orientation in the Ising and Potts models [8]). This is a consequence of the fact that a point on a grain boundary is believed to have a local velocity v given by $v\propto -1/R$, where R is the local radius of curvature [9]. [In this case, the left-hand side of Eq. (3) has dN/dt replaced by dA/dt.]

There is an important difference between bubbles on surfaces with negative K and those on surfaces with positive K. If $K>0$, stationary bubbles ($dN/dt=0$) exist for certain areas, provided that $n<6$. For example, on a sphere of radius R (large), bubbles with $n<6$ vertices whose areas are $\pi(6-n)R^2/3$ are stationary. When $K=0$, hexagons are stationary irrespective of size. For $K<0$ only bubbles with $n>6$ vertices are stationary, and the size and number of vertices are related.

An important consequence of Eq. (3) is that *no* bubble on a positively curved surface is *stable*, while all stationary bubbles on a surface with $K<0$ are stable. This is seen from Eq. (3) by perturbing off a stationary solution. If $K>0$, enlarging the bubble's area will cause gas to flow *into* the bubble, further enlarging it, while shrinking engenders further shrinkage. In contrast, a perturbation of a stationary bubble on a surface with $K<0$ will die out; the bubble will return to its original size at an *exponential* rate. Hexagons on a planar surface are the borderline case and are indifferent to changes in area.

The above observation implies that the ultimate fate of a two-dimensional foam on a positively curved surface is to evolve to one single bubble, covering the entire surface, while foam on a negatively curved surface will evolve to a final state where the bubbles have seven vertices or more, with each bubble having achieved its stable shape and size [10].

(2) A statistical property of 3D foams.—In the remainder of this paper, we shall be concerned with three-dimensional foams which are comprised of polyhedral-like bubbles. A three-dimensional polyhedral foam may be thought of as a collection of vertices, each of which is common to four edges, six faces, and four bubble cells. Each edge is common to three faces, and each face to two bubbles. The angle θ between any pair

of the four edges common to a vertex is the tetrahedral angle, $\theta \equiv \arccos(-1/3)$, and the dihedral angle between any two faces with a common edge is $2\pi/3$.

The von Neumann law does not have an analog in three dimensions in the sense that a soap bubble with sufficiently many faces is guaranteed to grow. (It is possible to construct counterexamples to this assertion [11].) Nevertheless, the tool that led to the von Neumann law in two dimensions, namely, the Gauss-Bonnet theorem, can be applied to a foam in three dimensions, and as we shall see it gives a relation and inequalities for static properties.

Consider a given bubble of the foam. The Gauss-Bonnet formula holds for each of its faces where now K is the Gaussian curvature of a face [12]. Each edge is common to two faces of the bubble, and the geodesic curvature C_g of the edge depends on the face being considered, since, although $d\mathbf{t}/dx$ is a property of the edge alone, $\boldsymbol{\gamma}$ points in different directions for different faces. Because of this, summing over the faces of a given bubble does not yield a simple relation. However, summing over all the bubbles in the foam leads to cancellations and to a simple relation involving average quantities.

To properly account for signs we note that Eq. (2) is invariant under change of orientation of the surface B. To properly account for multiplicities we note that each face is common to two bubbles, each edge to three bubbles, and each vertex to four bubbles. Therefore, each face appears twice in the sum, and each vertex twelve times (since for each bubble, it is common to three faces, and four bubbles meet at a vertex). Each edge must be counted twice for each of its three common faces. Combining all terms, we see that, up to boundary terms,

$$2\sum_{\text{faces}}\int_{\text{face}} K\,dA + 2\sum_{\text{edges}}\int_{\text{edge}} (\boldsymbol{\gamma}^1+\boldsymbol{\gamma}^2+\boldsymbol{\gamma}^3)\cdot\frac{d\mathbf{t}}{dX}\,dX + 12V(\pi-\theta) = 4\pi F\,, \tag{4}$$

where V and F are the total number of vertices and faces in the foam, respectively. But $\boldsymbol{\gamma}^1+\boldsymbol{\gamma}^2+\boldsymbol{\gamma}^3=0$ at each point along the edge, since the three vectors lie in a plane and the angle between each pair is $2\pi/3$. Next, denote the average value of $\int_{\text{face}}K\,dA$ by $\langle K\rangle$, and express V in terms of F. Since each vertex is common to six faces, if the average number of vertices *per face* is $\langle n\rangle$, then $V=\langle n\rangle F/6$. Canceling like factors we obtain our second main result:

$$\langle K\rangle + \langle n\rangle(\pi-\theta) = 2\pi\,. \tag{5}$$

This formula relates the average number of vertices per face to the average (integrated) curvature of the faces, and holds for all three-dimensional polyhedral foams in the limit where the surface-to-volume ratio is small [13].

Observations on a wide variety of foams yield values for $\langle n\rangle$ close to, but not precisely coinciding with, 5.104... [14]. This value has a history in statistical packing problems [15,16]. From Eq. (5) we see that it is obtained for $\langle K\rangle=0$. Since there are examples of foam with $\langle K\rangle<0$ this value does not hold in general. (In fact, we do not know if it ever holds.)

As an application of Eq. (5) which leads to an inequality on $\langle n\rangle$ consider the case of isobaric foam, i.e., foam where all bubbles have the same pressure. This is the case for periodic foam where a bubble is a unit cell. The equality of pressure of neighboring bubbles say that the soap films are minimal surfaces and have zero mean curvature. This implies that the Gaussian curvature is negative. It follows that for isobaric foam $\langle n\rangle > 5.104\ldots$. It is amusing that the periodic foam constructed by Lord Kelvin [17] has $\langle n\rangle = 5.143$, and so comes surprisingly close to, but still consistent with the general lower bound.

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[13] Note that this equation is distinct from the well-known relation $\bar{n}=6-12/\bar{f}$. Here, $\langle n\rangle=\sum_B\sum_F n(F)/\sum_B\sum_F 1$, where the first sum is over all the bubbles in the foam and the second over the faces of a given bubble, whereas $\bar{n}=V_T/F_T$, where V_T and F_T are the total number of vertices and faces in the foam, respectively. It is simple to show that $\langle n\rangle$ obeys $\langle n\rangle=6-12\langle 1/f\rangle$. For isobaric foam, this gives an *upper bound* $\langle n\rangle<6$, while Eq. (5) gives a *lower bound* $\langle n\rangle>5.104$.
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