Current Algebra in Three Dimensions

G. Ferretti and S. G. Rajeev
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 15 July 1992)

We study a three-dimensional analog of the Wess-Zumino-Witten model, which describes the Goldstone bosons of three-dimensional quantum chromodynamics. The topologically nontrivial term of the action can also be viewed as a nonlinear realization of Chern-Simons form. We obtain the current algebra of this model by canonical methods. This is a three-dimensional generalization of the Kac-Moody algebra.

PACS numbers: 11.40.Fy, 11.10.Lm

Three-dimensional field theories with topological terms in the action have been studied recently in many physical contexts, such as Chern-Simons theories [1], models for antiferromagnets [2], and anyon statistics [3] (see [4] for an overview). It is known that topological terms affect the statistics of solitons [3,5], and can change the canonical commutation relation of observables [6–8]. In three dimensions, this issue was studied in Refs. [9,10]. From another direction, three-dimensional nonlinear models have been shown to be renormalizable [11] in the $1/N$ expansion, although they are not so by power counting.

In this paper we study a three-dimensional nonlinear sigma model on a coset space with a topological term. It can be viewed as a three-dimensional analog of the Wess-Zumino-Witten [5] model or as a nonlinear Chern-Simons theory [1]. The model arises as the low-energy limit of three-dimensional quantum chromodynamics (QCD) [12]. It is also related to certain models for antiferromagnetism that arise in attempts to explain high-T_c superconductivity [2]. There are solitons in this model whose statistics is determined by the topological term. Furthermore, the theory should be renormalizable in the $1/N$ expansion. The focus of the present paper is the canonical formalism of this model. We discover by this method a three-dimensional generalization of the Kac-Moody algebra; it is a nontrivial Abelian extension of the naive current algebra. Like the Kac-Moody algebra, this can be further extended to a semidirect product with the algebra of vector fields.

The Wess-Zumino-Witten (WZW) model [5] in an even-dimensional space-time describes anomalous global symmetries. The field variable g takes values in a compact Lie group G [typically SU(N)]. It satisfies the classical equation of motion (for two dimensions):

$$\partial_\mu(g^{-1}\partial^\mu g)+\lambda\epsilon^{\mu\nu}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g)=0\,.$$

If $\lambda=0$, the equation is invariant under two discrete symmetries, $P_1: g\to g^{-1}$, and $P_2: g(t,x)\to g(t,-x)$. The WZW term breaks the symmetry down to the product P_1P_2. Thus, the WZW term forces the fundamental field of the theory to be a pseudoscalar. It is possible to formulate [5,6,13] this theory in a canonical formalism entirely in terms of currents. The classical Poisson brackets of the currents then define an infinite-dimensional Lie algebra, the current algebra. In the absence of the WZW term the current algebra is just the set of maps from space to the Lie algebra of G, with the pointwise bracket [6]. The current algebra is modified by the WZW term. In general, it provides an extension of the current algebra by an Abelian algebra [7,8]. In particular, in two-dimensional space-time, the current algebra is a central extension of the loop algebra, the well-known affine Kac-Moody algebra. The representation theory of this algebra is well understood. The relation to the WZW model has clarified the representation theory by relating it to conformal field theory.

Much less is known about the representation theory of current algebras in higher dimensions. Some progress has been made in this direction [14], although a complete understanding is still not available. This motivated us to look for an analog of the WZW model in 2+1 dimensions. This would be a way to study current algebras and boson-fermion equivalence in a context simpler than in 3+1 dimensions, yet more general than in 1+1 dimensions.

However, there are no anomalies for continuous symmetries in odd-dimensional space-time. This is related to the fact that $H^4(G)$ vanishes for the classical Lie groups. We can find an analog by looking for a nonlinear sigma model on a target space with H^4 nonzero. Furthermore, the additional term must preserve parity if the field variable is a pseudoscalar. The answer [12] is the target space $\mathrm{Gr}_{n,N}=\mathrm{U}(N)/\mathrm{U}(n)\times\mathrm{U}(N-n)$, the Grassmannian. We can parametrize $\mathrm{Gr}_{n,N}$ by an $N\times N$ matrix Φ:

$$\mathrm{Gr}_{n,N}=\{\Phi|\Phi^\dagger=\Phi;\ \Phi^2=1;\ \mathrm{tr}\Phi=N-2n\}\,.$$

The nonlinear model with this target space has the field equation $[\Phi,\partial_\mu\partial^\mu\Phi]=0$. The cohomology group $H^4(\mathrm{Gr}_{n,N})=Z\oplus Z$ (for $N\geq 4$, $n\geq 2$) is generated by ω_4 and $\omega_2\wedge\omega_2$, where $\omega_{2k}=\mathrm{tr}\Phi(d\Phi)^{2k}$. Of the two generators, only ω_4 is odd under the transformation $\Phi\to-\Phi$. Thus we arrive at a generalization of the WZW model to 2+1 dimensions,

$$F_\pi[\Phi,\partial_\mu\partial^\mu\Phi]+(k/8\pi)\epsilon^{\mu\nu\rho}\partial_\mu\Phi\,\partial_\nu\Phi\,\partial_\rho\Phi=0\,. \quad (1)$$

The coupling constant F_π has dimension of inverse length in the classical theory. This equation of motion follows from the multivalued action

$$S[\Phi]=\frac{F_\pi}{2}\int_M \mathrm{tr}\,d\Phi * d\Phi+\frac{k}{64\pi}\int_{M_4}\mathrm{tr}\,\Phi(d\Phi)^4 . \quad (2)$$

Here, M_4 is a four-dimensional manifold whose boundary is space-time M. As in the WZW model, in order that $\exp(iS[\Phi])$ be independent of the continuation into the fourth dimension, k must be an integer. This theory is invariant under parity (with Φ transforming as a pseudoscalar) if $N=2n$. In this case it is the low-energy limit of three-dimensional QCD with an even number $2n$ of flavors and k colors [12]. But we will study this theory for general N and n.

We will now present a canonical formulation of this theory, in terms of a set of Poisson brackets for the basic observables, a set of first-class constraints, and a Hamiltonian. The Poisson brackets of the theory will be a generalization of the Kac-Moody algebra to 2+1 dimensions. The canonical formalism is in terms of a decomposition of space-time $M=\Sigma\times R$, Σ being the two-dimensional surface at fixed time. It is possible to derive this formalism from an action principle, but the appropriate one is not the multivalued action (2). Define a new variable g valued in $G\equiv \mathrm{U}(N)$. One can always write $\Phi=g\epsilon g^\dagger$ with $\epsilon=\mathrm{diag}\{1,\ldots,1,-1,\ldots,-1\}$, and $\mathrm{tr}\,\epsilon=N-2n$. Then Φ is invariant under right multiplication of g by elements that commute with ϵ; i.e., under $g\mapsto gh$, $h\in H\equiv \mathrm{U}(n)\times\mathrm{U}(N-n)$. These transformations are therefore like gauge transformations and we can write an action for the theory in terms of g that respects this gauge invariance:

$$S[g,A]=-2F_\pi\int_M \mathrm{tr}[(g^\dagger dg-A)*(g^\dagger dg-A)]-\frac{k}{8\pi}\int_M \mathrm{tr}[\epsilon(g^\dagger dg)^3-\tfrac{1}{3}(g^\dagger dg\,\epsilon)^3]\,. \quad (3)$$

The one-form A is an auxiliary gauge field valued in $\underline{H}$, the Lie algebra of H. Its purpose is to remove the unwanted degrees of freedom. In this form of the action, the topologically nontrivial term is a nonlinear realization of the Chern-Simons term. Unlike (2), action (3) is a local integral on space-time but it is only gauge invariant up to a multiple of 2π.

It is now possible to derive Poisson brackets and constraints from this action by a conventional procedure. The Poisson brackets so obtained at first will involve nonlinear (cubic) terms in Φ. However, we can remove these by appropriate redefinition of the generators. The variables Φ,J that satisfy simple commutation relations are, in this language, $\Phi=g\epsilon g^\dagger$ and

$$J=g\left(F_\pi R+\frac{k}{16\pi}\epsilon^{ij}(g^\dagger\partial_i g\,g^\dagger\partial_j g\,\epsilon+\epsilon g^\dagger\partial_i g\,g^\dagger\partial_j g-\epsilon g^\dagger\partial_i g\,\epsilon g^\dagger\partial_j g\,\epsilon)\right)g^\dagger .$$

Here, R is the projection of $g^\dagger\dot g$ on the orthogonal complement of the gauge group and must satisfy the constraint $[R,\epsilon]_+=0$. (The symbol $[\,]_+$ will denote the anticommutator throughout the paper.) Our conventions are that Φ is Hermitian and J anti-Hermitian.

We present only the results of the canonical analysis, leaving the details for a longer publication. The basic observables of the theory are Φ and J (which is essentially the time component of the current) specified on Σ. It is natural to think of Φ as a scalar on Σ and of J as a scalar density (two-form). Let us also introduce the test functions $\lambda:\Sigma\to\underline{G}$ scalar, and $\xi:\Sigma\to\underline{G}$ scalar density. ($\underline{G}$ denotes the Lie algebra of G.) We give the Poisson brackets in terms of the dimensionless quantities $\Phi(\xi)=\int_\Sigma \mathrm{tr}(\Phi\xi)d^2x$ and $J(\lambda)\equiv\int_\Sigma \mathrm{tr}(J\lambda)d^2x$. They are

$$\{\Phi(\xi),\Phi(\xi')\}=0,\quad \{J(\lambda),\Phi(\xi)\}=\Phi([\lambda,\xi])\,,$$
$$\{J(\lambda),J(\lambda')\}=J([\lambda,\lambda'])+k\Phi(\omega(\lambda,\lambda'))\,. \quad (4)$$

In (4), ω is defined as $\omega(\lambda,\lambda')=(1/16\pi)\epsilon^{ij}[\partial_i\lambda,\partial_j\lambda']_+$. If the space Σ is a torus, we can write these relations more explicitly in a plane-wave basis:

$$\{\Phi^a_m,\Phi^b_n\}=0,\quad \{J^a_m,\Phi^b_n\}=f^{abc}\Phi^c_{m+n}\,,$$
$$\{J^a_m,J^b_n\}=f^{abc}J^c_{m+n}-\frac{k}{16\pi}d^{abc}\epsilon^{ij}m_i n_j\Phi^c_{m+n}\,. \quad (5)$$

In (5), m,n are two-dimensional vectors with integer components. Also, d^{abc} is the usual symmetric cubic invariant of $\mathrm{U}(N)$ and f^{abc} the structure constants.

The algebra has to be supplemented by two constraints,

$$\Phi^2=1 \text{ and } [J,\Phi]_++\frac{k}{16\pi}\epsilon^{ij}(\partial_i\Phi\,\partial_j\Phi)=0\,. \quad (6)$$

One can verify that these are first-class constraints. This is one major difference between our treatment of the problem and the usual canonical formalism for similar models [9]. With our method, second-class constraints never arise and there is no need to introduce Dirac brackets. Actually, our constraints (6) satisfy even stronger relations than the conditions for being first class. It can be easily checked from (4) that the Poisson brackets of Φ and J with (6) vanish weakly. This means that every two weakly equivalent observables $A\approx B$ of the theory will have weakly equivalent Poisson brackets with any third observable C: $\{A,C\}\approx\{B,C\}$. Both constraints and Poisson brackets are also invariant under diffeomorphisms of Σ.

The canonical formalism is completed by the Hamiltonian

$$H=\frac{1}{2}\int_\Sigma \mathrm{tr}\left[-\frac{1}{F_\pi\sqrt{g}}\left(J+\frac{k}{32\pi}\epsilon^{ij}\partial_i\Phi\partial_j\Phi\right)^2+\frac{F_\pi\sqrt{g}}{4}g^{ij}\partial_i\Phi\partial_j\Phi\right]d^2x\,. \tag{7}$$

Of course, the Hamiltonian does depend on the metric g_{ij} on Σ. The equations of motion that follow are

$$\dot\Phi=(1/F_\pi\sqrt{g})[J,\Phi]\,,$$
$$\dot J=-\frac{1}{4}F_\pi\sqrt{g}g^{ij}[\Phi,\partial_i\partial_j\Phi]+\frac{k}{32\pi F_\pi\sqrt{g}}\epsilon^{ij}\{[\partial_iJ,\partial_j\Phi]_+-\partial_i(\Phi J\Phi)\partial_j\Phi-\partial_j\Phi\partial_i(\Phi J\Phi)\}\,. \tag{8}$$

Equation (1) for Φ then follows (in flat space) from this system of first-order equations and the constraints. This completes our discussion of the canonical formalism.

Equations (4) [or (5)] define our current algebra $\mathcal{G}_k$ (more exactly a current-field algebra). Notice that the Poisson brackets yield *linear* relations in Φ and J, so (4) defines a Lie algebra. If we set $k=0$, the J's alone form a subalgebra $\mathcal{J}$. The Φ's generate an Abelian subalgebra $\mathcal{V}$ of $\underline{G}$-valued densities. The vector space $\mathcal{V}$ can be identified with dual of the Lie algebra $\mathcal{J}$ by the natural pairing $\langle\lambda,\xi\rangle=\int_\Sigma \mathrm{tr}(\lambda\xi)d^2x$, so that it carries the coadjoint representation of $\mathcal{J}$. When $k=0$, our algebra $\mathcal{G}_0$ reduces to the semidirect product of $\mathcal{J}$ with its coadjoint representation.

When $k\neq0$, $\mathcal{G}_k$ is an Abelian extension of the map algebra $\mathcal{J}$ by its coadjoint representation. The Jacobi identity of $\mathcal{G}_k$ is equivalent to the statement that $\omega:\mathcal{J}\wedge\mathcal{J}\to\mathcal{V}$ is a two-cocycle of the Lie algebra cohomology:

$$\partial\omega(\lambda_1,\lambda_2,\lambda_3)\equiv[\lambda_1,\omega(\lambda_2,\lambda_3)]+\omega(\lambda_1,[\lambda_2,\lambda_3])+\text{cyclic}=0\,, \tag{9}$$

which can be verified by direct computation. If ω had been exact there would have been a linear function $\mu:\mathcal{J}\to\mathcal{V}$ such that

$$\omega(\lambda,\lambda')=\partial\mu(\lambda,\lambda')\equiv-\mu([\lambda,\lambda'])+[\lambda,\mu(\lambda')]-[\lambda',\mu(\lambda)]\,. \tag{10}$$

There is no such μ, so we cannot reduce our algebra to a semidirect product by a change of basis; $\mathcal{G}_k$ is a nontrivial Abelian extension of $\mathcal{J}$ by $\mathcal{V}$.

It is useful to note that the above extension can be "exponentiated" to an extension of the group of maps $g:\Sigma\to G$ by the vector space $\mathcal{V}$ (thought of as an Abelian group). The multiplication law $\circ$ is

$$(g_1,\xi_1)\circ(g_2,\xi_2)=(g_1g_2,\xi_1+g_1\xi_2g_1^{-1}+k\Omega(g_1,g_2))\,, \tag{11}$$

where $\Omega(g_1,g_2)=(1/16\pi)\epsilon^{ij}\partial_ig_1\partial_jg_2g_2^{-1}g_1^{-1}$. The associativity of the group multiplication $\circ$ requires that Ω be a group two-cocycle [15]:

$$\partial\Omega(g_1,g_2,g_3)\equiv-\Omega(g_1g_2,g_3)+\Omega(g_1,g_2g_3)-\Omega(g_1,g_2)+g_1\Omega(g_2,g_3)g_1^{-1}=0\,. \tag{12}$$

This identity can be proved by direct computation. It is possible to understand the constraints (6) as describing a coadjoint orbit of the above group. The symplectic form and hence the part of the action that is linear in time derivatives can be understood from Kirillov's method of orbits applied to this case. In fact, it turns out that the Poisson bracket structure derived in this way coincides with (4).

Finally, recall that the Kac-Moody algebra is invariant under the action of the Virasoro algebra. In fact the generators of the Virasoro algebra can be written in terms of the currents. The analog of the Virasoro algebra in our case is the algebra of vector fields on Σ. The Lie algebra $\mathcal{G}_k$ is invariant under diffeomorphisms, so that it can be extended as a semidirect product with the algebra of vector fields on Σ. If u and v are such vector fields, and L is the generator associated to them, satisfying $\{L(u),L(v)\}=L([u,v])$, then

$$\{L(u),J(\lambda)\}=J(u^i\partial_i\lambda),\quad\{L(u),\Phi(\xi)\}=\Phi(\partial_i(u^i\xi))\,. \tag{13}$$

It would be interesting to develop a representation theory for the algebra (5). Physically, that would correspond to quantizing the above field theory. This might look impossible at first because the theory is not renormalizable by power counting. However, as remarked in [12], the theory is renormalizable in the $1/N$ expansion, provided one allows for massive vector fields to be dynamically generated. We would first consider the limit $N\to\infty$ (keeping n and k fixed) that is solvable by the saddle-point method. This model has a nontrivial UV fixed point. This is the analog of the UV fixed point of the WZW model in 1+1 dimensions (although in that case the UV fixed point is trivial). The WZW model also has a nontrivial IR stable fixed point. It is possible that there is an analogous (*nontrivial*) IR fixed point in our theory as well.

Another interesting issue is that of the fermion-boson correspondence. In two dimensions the WZW model, at the IR fixed point, corresponds to a free Fermi theory. There has already been an attempt to prove such an equivalence for the CP^1 model [16]. However, there is, at present, no reliable approximation method to study this issue, because the correspondence breaks down for CP^N with $N > 1$. Our model should have fermionic equivalents for any N, n, so that the issue can be studied within the $1/N$ expansion. We will report on work in this direction in a later publication.

We would like to thank A. P. Balachandran, P. Teotonio, and Z. Yang for discussions. This work was supported in part by the U.S. Department of Energy, Contract No. DE-AC02-76ER13065.

[1] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982); J. Schonfeld, Nucl. Phys. **B185**, 157 (1981); E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[2] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, New York, 1991); P. B. Wiegmann, Phys. Scr. **T27**, 160 (1989); N. Read and S. Sachdev, Nucl. Phys. **B316**, 609 (1989).

[3] F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).

[4] S. Forte, Rev. Mod. Phys. **64**, 193 (1992); A. P. Balachandran, M. Bourdeau, and S. Jo, Int. J. Mod. Phys. A **5**, 2423 (1990).

[5] E. Witten, Nucl. Phys. **B223**, 422 (1983); Commun. Math. Phys. **92**, 455 (1984).

[6] S. G. Rajeev, Phys. Rev. D **29**, 2944 (1984).

[7] J. Mickelsson, Commun. Math. Phys. **97**, 361 (1985).

[8] L. D. Faddeev, Phys. Lett. **145B**, 81 (1984).

[9] M. J. Bowick, D. Karabali, and L. C. R. Wijewardhana, Nucl. Phys. **B271**, 417 (1986).

[10] G. Semenoff, P. Sodano, and Y. S. Wu, Phys. Rev. Lett. **62**, 715 (1989); G. Semenoff, Phys. Rev. Lett. **61**, 517 (1988); R. Banerjee, Phys. Rev. Lett. **69**, 17 (1992).

[11] I. Y. Aref'eva and S. I. Azakov, Nucl. Phys. **B162**, 298 (1980); B. Rosenstein, B. J. Warr, and S. H. Park, Nucl. Phys. **B336**, 435 (1990); A. Kovner and B. Rosenstein, Phys. Lett. B **261**, 104 (1990).

[12] G. Ferretti, S. G. Rajeev, and Z. Yang, University of Rochester Report No. UR 1255, 1992 [Int. J. Mod. Phys. (to be published)]; University of Rochester Report No. UR 1256 [Int. J. Mod. Phys. (to be published)].

[13] G. Bhattacharya and S. G. Rajeev, Nucl. Phys. **B246**, 157 (1984).

[14] J. Mickelsson and S. G. Rajeev, Commun. Math. Phys. **116**, 365 (1988); Lett. Math. Phys. **21**, 173 (1991).

[15] J. Mickelsson, *Current Algebras and Groups* (Plenum, New York, 1989).

[16] A. Kovner, Int. J. Mod. Phys. A **5**, 3999 (1990).