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Yangian Symmetry of Integrable Quantum Chains with Long-Range Interactions and a New Description of States in Conformal Field Theory

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The SU(n) quantum chains with inverse-square exchange exhibit a novel form of Yangian symmetry compatible with periodic boundary conditions, allowing states to be countable. We characterize the "supermultiplets" of the spectrum in terms of generalized "occupation numbers." We embed the model in the k=1 SU(n) Kac-Moody algebra and obtain a new classification of the states of conformal field theory, adapted to particlelike elementary excitations obeying fractional statistics.

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In this Letter, we introduce a new description of the states of non-Abelian conformal field theories that is in some sense a generalization of the Fock-space occupation-number description to describe excitations of an ideal gas with fractional statistics [1]. This is applicable to, e.g., the "spinon" excitations in a gapless Fermi fluid (Luttinger liquid) with spin-charge separation, in one or possibly higher spatial dimension. The results also appear to shed new light on the algebraic structures of integrable models, placing the Bethe-ansatz-solvable models in the context of a larger family of models, and identifying the inverse-square interaction models as possibly the simplest example of Yang-Baxter integrability, where the excitations have purely statistical interactions.

These results have emerged from an extensive study by three of us [2] of the remarkable underlying symmetry algebra of the $S=\frac{1}{2}$ Heisenberg spin chains with inverse-square interactions [3,4], and an embedding of this model and its SU(n) generalizations in conformal field theory. As that study reached completion, it became apparent that the algebraic structures under investigation were a novel presentation of the Yangian algebra [5] that has been emphasized by another of us [6] as the key algebraic structure in integrable models with non-Abelian symmetry. The novel feature is that in contrast to the usual presentation (e.g., in Bethe-ansatz models [7]), this form of the Yangian can coexist with periodic boundary conditions that make states countable.

We will first describe the integrable spin-chain Hamiltonians

$$H_2 = \sum_{ij}{}' \left(\frac{z_i z_j}{z_{ij} z_{ji}}\right)(P_{ij}-1), \tag{1}$$

where P_{ij} exchanges the states on sites i and j. The primed sum omits equal values of the summation variables, and $z_{ij} \equiv z_i - z_j$. The distinct complex numbers z_i parametrize the lattice sites. Translational invariance is present if $\{z_i\} = \{\omega^n\}$; inversion symmetry ($z_i \to 1/z_i$) means we can choose $|\omega| \leq 1$.

There are two families of models where H_2 is Hermitian and translationally invariant. The *trigonometric* models have $\omega = \exp(2\pi i/N)$, and N distinct sites on a circle, with exchange between sites proportional to the inverse-square of their chord distance. With two states per site [SU(2) or $S=\frac{1}{2}$ Heisenberg chain] this model was independently introduced in [3] and [4], and has a straightforward SU(n) generalization [1, 8] to n states per site. The *hyperbolic* family has $N=\infty$, and real ω in the ranges $[-1,1]$. Rescaling H_2 in the singular limit $\omega \to 0$ gives the familiar nearest-neighbor-exchange Bethe-ansatz (BA) model [9, 10]. The rescaled limits of the hyperbolic and trigonometric models as $\omega \to 1$ coincide in the inverse-square-exchange (ISE) model; for $\omega \to -1$ the hyperbolic model scales to two decoupled inverse-square models. The $n = 2$ positive-ω hyperbolic family (inverse-sinh^2 exchange), was recently proposed [11] as the natural integrable interpolation between the BA and ISE $S = \frac{1}{2}$ Heisenberg chains.

Let $\mathbf{X}_i$ be the $n \times n$ operator matrix with elements $X_i^{\alpha\beta}$ that act as $|\beta\rangle\langle\alpha|$ on site i. Then spins $J_i^a = \mathrm{Tr}(\mathbf{t}^a \mathbf{X}_i)$ are defined using the $n^2 - 1$ traceless Hermitian matrices $\mathbf{t}^a$ of the fundamental representation of the generators of SU(n), normalized so $\mathrm{Tr}(\mathbf{t}^a\mathbf{t}^b) = \frac{1}{2}\delta^{ab}$. These obey the Lie algebra $[\mathbf{t}^a, \mathbf{t}^b] = f^{abc}\mathbf{t}^c$, with antisymmetric structure constants $f^{abc} = c_v f_{abc}$, $f^{abc} f_{dcb} = \delta_d^a$, and $c_v = n$. For $n = 2$, $f^{abc} = i\epsilon^{abc}$; $2\mathbf{t}^a$ are Pauli matrices.

We will consider the algebra derived from supplementing the usual global SU(n) generators $Q_0^a = \sum_i J_i^a$ ("level 0") with "level-1" generators

$$Q_1^a(\{z_i\}) = \tfrac{1}{2}{\sum_{ij}}' w_{ij} f^{abc} J_i^b J_j^c, \tag{2}$$

where $w_{ij} = (z_i + z_j)/z_{ij}$ is the general solution of the condition $\Delta_{ijk} \equiv w_{ij}w_{ik} + w_{jk}w_{ji} + w_{ki}w_{kj} = 1$ for $i \neq j \neq k$. The only condition on the $\{z_i\}$ is that they be distinct.

Drinfel'd [5] has defined the Yangian (a Hopf algebra) by starting with level-0 and level-1 generators, recursively defining level-n generators by $f^{abc}[Q_1^c, Q_{n-1}^b] = c_v Q_n^a$, and demanding consistency at level n, i.e., that there is only *one* independent generator Q_n^a at each level, regardless of the sequence of commutations by which it was obtained. Provided the conditions at levels 2 and 3 are satisfied, so are all the higher ones. If $\Gamma_2^{abc} = [Q_1^a, [Q_1^b, Q_0^c]]$, level-2 consistency requires that $\Gamma_2^{abc} + \Gamma_2^{bca} + \Gamma_2^{cab}$ equals

$$\tfrac{1}{4} f^{akr} f^{bls} f^{cmt} f_{rst} \{Q_0^k, Q_0^l, Q_0^m\}, \tag{3}$$

where the symmetrized product $\{A_1, A_2, \ldots, A_k\}$ is the average over all $k!$ orderings of the operators. If $\Gamma_3^{abcd} = [[Q_1^a, Q_1^b], [Q_1^c, Q_0^d]]$, level-3 consistency similarly requires that $\Gamma_3^{abcd} + \Gamma_3^{dcba}$ can be expressed in terms of $\{Q_1^k, Q_0^l, Q_0^m\}$. The fundamental property of the Yangian is that it has the homomorphism $Q_n^a \to \Delta(Q_n^a)$ where the *comultiplication* $\Delta(Q_n^a)$ acts on two copies of the Hilbert space: as usual, $\Delta(Q_0^a) = \mathbb{1} \otimes Q_0^a + Q_0^a \otimes \mathbb{1}$, but $\Delta(Q_1^a)$ nontrivially couples the two copies:

$$\Delta(Q_1^a) = \mathbb{1} \otimes Q_1^a + Q_1^a \otimes \mathbb{1} + \tfrac{1}{2} f^{abc} Q_0^b \otimes Q_0^c. \tag{4}$$

Other homomorphisms are $Q_n^a \to (-1)^n Q_n^a$ and $Q_n^a \to Q_n^a(\lambda)$, where λ is a "spectral parameter," and $Q_1^a(\lambda) = Q_1^a + \lambda Q_0^a$. If $\{H_0^i, E_0^\alpha\}$ is the SU(n) Cartan basis, then Yangian highest weight states satisfy $E_n^\alpha|\Psi\rangle = 0$ for $\alpha > 0$, and are eigenstates of the $\{H_n^i\}$. The eigenvalues determine certain polynomials; their roots characterize the representation of the Yangian [12].

We verified the two consistency conditions, demonstrating that recursive action of $Q_1^a(\{z_i\})$ generates the Yangian. The two key ingredients in the calculation are (a) $\Delta_{ijk} = 1$ for $i \neq j \neq k$, and (b) that J_i^a acts in the fundamental representation, which eliminates problem terms where i, j, and k are not distinct. It is also found that the requirements for Q_1^a to commute with $H_2 = \sum_{i<j} h_{ij} P_{ij}$ are that H_2 is a multiple of (1), and $\{z_i\} = \{\omega^n\}$. The Yangian generator in the BA limit $\omega \to 0$ has been described previously [7].

We note that Q_1^a *does not commute with the Casimirs of the group*. This explains why the eigenfunctions of the trigonometric models generally form *highly reducible* representations of the group (the "supermultiplet" structure [3]). Note also that Q_1^a is odd parity under inversion.

As the models are integrable, commuting Hamiltonian constants of the motion H_n ($n \geq 2$) are in principle obtained by the Taylor expansion of the logarithm of $T(\lambda)$, the trace of an operator-valued *monodromy matrix* $L^{\alpha\beta}(\lambda)$ [13], with commutation relations defined by a rational solution of the Yang-Baxter equation $R_{12}^{\alpha\gamma,\beta\delta} = (\lambda_1 - \lambda_2)\delta^{\alpha\beta}\delta^{\gamma\delta} + i\delta^{\alpha\delta}\delta^{\gamma\beta}$. If $L_1^{\alpha\gamma,\beta\delta} \equiv L^{\alpha\beta}(\lambda_1)\delta^{\gamma\delta}$ and $L_2^{\alpha\gamma,\beta\delta} \equiv \delta^{\alpha\beta} L^{\gamma\delta}(\lambda_2)$, then $\mathbf{R}_{12}\mathbf{L}_1\mathbf{L}_2 = \mathbf{L}_2\mathbf{L}_1\mathbf{R}_{12}$, so $[T(\lambda), T(\lambda')] = 0$. Unfortunately, we have not yet obtained $L^{\alpha\beta}(\lambda; \{z_i\})$ away from the BA limit.

The H_n are *extensive*, with parity $(-1)^n$, and $[H_m, H_n] = [Q_m^a, H_n] = 0$. They must coincide with the known results in the BA limit. H_3 is given by [11]

$$H_3 = {\sum_{ijk}}' \left(\frac{z_i z_j z_k}{z_{ij} z_{jk} z_{ki}}\right)(P_{ijk} - 1). \tag{5}$$

Proceeding empirically, we obtained the next term

$$H_4 = {\sum_{ijkl}}' \left(\frac{z_i z_j z_k z_l}{z_{ij} z_{jk} z_{kl} z_{li}}\right)(P_{ijkl} - 1) + H_4', \tag{6}$$

where the "obvious" guess is supplemented by

$$H_4' = -\tfrac{1}{3} H_2 - 2{\sum_{ij}}' \left(\frac{z_i z_j}{z_{ij} z_{ji}}\right)^2 (P_{ij} - 1). \tag{7}$$

The commutation of H_4 with H_2, H_3, and Q_1^a was confirmed numerically [2] using matrix representations of dimensions up to 100 numerically constructed on symmetry subspaces of the states of finite-length chains. The numerical study also allowed us [2] to empirically characterize the eigenvalue spectrum, and generalize the results previously obtained for the SU(2) case [14].

The eigenfunctions of the trigonometric models are grouped into multiplets characterized by sets of distinct integer "*rapidities*" $\{m_i\}$, in the range $0 < m_i < N$:

$$H_n|\{m_i\}, \mu\rangle = \sum_i \epsilon_n(m_i)|\{m_i\}, \mu\rangle. \tag{8}$$

In particular, $\epsilon_2(m) = m(m - N)$, $\epsilon_3(m) = \frac{1}{2}\epsilon_2(m)\epsilon_2'(m)$, and $\epsilon_4(m) = \epsilon_2(m)[\frac{1}{6}N^2 + \epsilon_2(m)]$. The crystal momentum of the state is given by $K = (2\pi i J/N) \pmod{2\pi}$, where $J = \sum_i m_i$.

A simple rule gives the complete set of eigenvalues: If the group is SU(n), *all sequences* $\{m_i\}$ *not containing* n *or more consecutive integers occur.* The rapidity sequence can also be represented by a sequence of $N+1$ integers (representing the range $0, 1, \ldots, N$) taking values 0 or 1; the 1's indicate the integers (*rapidities*) in the set $\{m_i\}$ (the first and last entries are 0). For example, the singlet ground state of the SU(3) chain with $N = 9$ is represented by 0110110110. This is the *only* state of this chain which is singlet under the action of the Yangian generators Q_n^a.

To obtain the multiplicity, the rapidity sequence is transformed into a product of "*motifs*": First identify all occurrences of groups of $n-1$ consecutive 1's, and replace the 0's that enclose them by ")(" [e.g.,"0110" → ")(11)(" for SU(3)]. Then replace all remaining occurrences of the sequence "101" by "1)(1". Finally, replace the 0's at positions 0 and N by "(" and ")". Counting each ")" and "(" as half a symbol, a state is now represented by N symbols, organized into a sequence of *motifs*, delimited by parentheses. The special motif (11...1), with a symbol count of n, will be called the SU(n) *vacuum motif.* This represents a singlet combination of n spins belonging to the fundamental representation. The SU(n) ground state on nM sites is represented by a product of M singlet vacuum motifs; the $N = 9$ SU(3) ground state becomes (11)(11)(11).

A general rapidity sequence factors into a product of both vacuum motifs and *excited motifs.* The empirically observed principle is that each motif is associated with a definite SU(n) representation content, and *the total representation content of a state is the direct product of that of the constituent motifs.* An important class of motifs is the sequence ( ), (1), (11), ..., terminating with the vacuum motif. We will call these the *primary motifs.* Such motifs with symbol count r contain a single irreducible SU(n) representation with Young diagram $[1^r]$.

More general motifs (A0B) can be built from constituents (A)(B) by a "*fusion*" process that is essentially comultiplication. The direct product of representation contents of (A) and (B) is an *upper bound* to that of (A0B): Typically, some representations in the direct product are lost. For example, ( )( ) is represented by the Young diagram product $[1]\otimes[1] = [2]\oplus[1^2]$, but (0) contains only the symmetric combination [2]. Mirror-image motifs have the same SU(n) representation content.

A motif with symbol count r can be identified and characterized by studying the states of a chain of length r. The motifs (00...0) with symbol count r (containing only 0's) can be identified with the "ferromagnetic" (symmetric representation $[r]$) state of the chain. We call these the *symmetric motifs.* These are the only excited motifs in the simple SU(2) case, and were interpreted in [1] as describing r "spinons" in the same "orbital."

A systematic procedure that correctly determines the representation content of any given motif has been obtained [15] from a singular limit of the thermodynamic Bethe-ansatz (TBA) equations based on the "rapidity string hypothesis," thus providing the *fusion rules.*

The solutions of the TBA equations are in one-to-one correspondence with the true solutions of the BA equations, and give a correct description of the representation content of the states of a finite integrable chain solved by the BA, even though the complex rapidity patterns in the exact solutions show a rather deformed string structure. The cause of this deformation appears to be the violation of Yangian symmetry by the boundary conditions. It is not yet known how to obtain the rapidities for the finite periodic versions of the hyperbolic models [11], but we predict [15] that the deviations of the rapidity patterns from the ideal strings will continuously decrease to zero in the interpolation between the periodic BA and ISE models.

In the ISE limit, the integer rapidities in a motif represent complex rapidity strings which (after rescaling rapidities to keep their real parts finite) have collapsed to a single point on the real axis, so the distinction between one string, two strings, etc., is lost. The TBA equations determine the number of distinct ways that string lengths can be assigned to the rapidities; the distinct solutions count the irreducible SU(n) representations contained in the motif.

The fusion rules seem to represent *statistical interactions* between excitations [1]. The Pauli principle provides a prototype example of such effects, in that if a "coproduct" of the spin states of two electrons is made, the triplet spin combination is "missing" if they are in the *same* orbital. This suggests that the "motifs" are the fractional-statistics generalization of "independent orbitals" in the conventional ideal Fermi or Bose gases. The different possible motifs would then correspond to the different possible occupation states of an orbital. The coexistence in the trigonometric models of Yangian symmetry and periodic boundary conditions *that make states countable* is crucial in exposing these effects.

We finally show how the remarkable structures of the trigonometric lattice models are also present in conformal field theory (CFT) [2]. The derivation is heuristic, motivated by the observation that in the calculation of the trigonometric model correlation functions, the lattice sums over the z_i can be replaced by what is effectively a contour integral around the unit circle [3]. As given, it turns out to work only for the SU(2) case, but we expect a less-heuristic approach will generalize.

We introduce a field $J^a(z)$ with a mode expansion

$$J^a(z) = \sum_m J_m^a z^{-(m+1)}, \tag{9}$$

so $Q_0^a = J_0^a$, and construct

$$Q_1^a = P \oint \frac{dz_1}{2\pi i} \oint \frac{dz_2}{2\pi i} \left(\frac{z_1+z_2}{z_{12}}\right) f^{abc} : J^b(z_1) J^c(z_2) : ,$$
$$H_2 = P \oint \frac{dz_1}{2\pi i} \oint \frac{dz_2}{2\pi i} \left(\frac{z_1 z_2}{z_{12} z_{21}}\right) : J^a(z_1) J^a(z_2) : . \tag{10}$$

The exclusion of terms with $z_i = z_j$ in the sum is implemented by taking the principal part of the integral; the standard radial normal ordering is indicated. The J_m^a are interpreted as generators of the Kac-Moody algebra: $[J_m^a, J_n^b] = f^{abc}J_{m+n}^c + \frac{1}{2}km\delta_{m+n,0}\delta^{ab}$. It is found that the *central extension* must have $k = 1$, which corresponds to the CFT describing the low-energy fixed point of the corresponding spin chains [3, 4, 8].

In terms of the modes,

$$Q_1^a = f^{abc} \sum_m \mathrm{sgn}(m) : J_{-m}^b J_m^c : . \tag{11}$$

$$H_2 = \sum_m |m| : J_{-m}^a J_m^a : . \tag{12}$$

These commute with the Virasoro level operator L_0, where $(n+1)L_0 = \sum_m : J_{-m}^a J_m^a :$, which is the analog of crystal momentum. H_2 represents nonlocal (inverse-square interaction) corrections to the CFT that break Lorentz and conformal invariance.

The validity of this ansatz [for the $k = 1$ SU(2) case only] was confirmed by numerical diagonalization at fixed $L_0 \leq 10$. Q_1^a commutes with H_2, which has an essentially identical eigenvalue spectrum to the lattice model, except that the product of motifs is now semi-infinite. The vacuum ($L_0 = 0$) is represented by (1)(1)(1)... and the $S = \frac{1}{2}$ primary state ($L_0 = \frac{1}{4}$) by ( )(1)(1).... Excited states are given by a *finite* rearrangement of a primary-state rapidity sequence: $\{m_i^0\} \to \{m_i\}$. The eigenvalue of H_2 is given by $[\sum_i (m_i^0)^2] - [\sum_i (m_i)^2]$, and the value of L_0 relative to the primary state is $(\sum_i m_i^0) - (\sum_i m_i)$. The primary sector to which states belong is easily determined from the asymptotic phase of the product of vacuum motifs.

For SU(n) with $n > 2$, the ansatz (11) does not work, but we expect that operators H_2 and Q_1^a with analogous properties to those of the spin chains do exist, and hope to report these elsewhere. We assume that their eigenvalue spectra are related to the corresponding SU(n) chain in the same way as we found in the SU(2) case. Note that H_2 will break the conjugation invariance of the CFT [e.g., the symmetry between 3 and $\bar{3}$ excitations for SU(3)].

The finite subspace of H_2 eigenstates $|\Psi\rangle$ of the CFT that correspond to eigenstates of the trigonometric chain of length N are those satisfying the selection rule that the rapidity sequence has an unbroken sequence of vacuum motifs from position N upwards. A special class of "$W^{(n)}$-primary" states can be identified which are the highest weight states of Yangian multiplets with motif sequences like (00)(1)(1)(11)(11)... (the primary motifs only occur in order of their symbol count, following any symmetric motif). These (and their generalizations which also have similar motif patterns in reverse terminating at position N) correspond to the Gutzwiller or Jastrow states explicitly constructed as wave functions of the trigonometric chains [3, 4, 8].

These wave functions [1] are strikingly equivalent to correlation functions ("conformal blocks") of the CFT, leading to an obvious conjecture for a general relation between H_2 eigenstates $|\Psi\rangle$ of the field theory, and wave functions of the spin chain: $\Psi(\{z_i, \sigma_i\}) \propto \langle\{z_i, \sigma_i\}|\Psi\rangle$, where

$$|\{z_i, \sigma_i\}\rangle = \prod_i \phi(z_i, \sigma_i)|0\rangle. \tag{13}$$

Here primary fields $\phi(z, \sigma)$ (of the fundamental representation) act on the vacuum of the CFT.

We also note that systematic techniques for the explicit calculation of the thermodynamic limit of correlation functions of the Gutzwiller states have been successfully developed [16], with simple but nontrivial results. If these techniques can be reinterpreted in terms of Yangian symmetry, then smoothly deformed from the ISE model to the hyperbolic family, the problem of calculating integrable lattice model correlation functions would be solved.

We finally note that while we just studied the fundamental representation SU(n) chains and the corresponding $k = 1$ CFT, we expect that these results can be generalized by the technique of local symmetrization (fusion) of coproducts of k independent copies of the models. We also expect there to be generalizations to analogs of the XXZ chain where the SU(n) Yangian symmetry is deformed into that of the corresponding quantum group [7]. Indeed, it is conceivable that *all* BA lattice models have integrable generalizations of the type discussed here.

Models of the trigonometric type combine a generalization of the exact additivity of quasiparticle energies of the ideal Fermi and Bose gases, with the "generalized Pauli principle" introduced in [1] as an alternative to the braiding description of fractional statistics. The realization of quantum group algebra in a form compatible with periodic boundary conditions is likely to be at the heart of an "occupation-number" description of the "ideal gas" limit of systems of particles with fractional statistics. Applications to the quantum Hall effect may be anticipated.

The $k = 1$ conformal field theories studied here are obtained from free fermions by factoring out the charge degrees of freedom; for spin-$\frac{1}{2}$ electrons, the charge degrees of freedom are *also* described by the SU(2) conformal field theory. The Yangian basis described here is thus the natural one for describing semionic *spinon* and *holon* states in spin-charge separated systems.

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