A New Class of Defects

Tanmay Vachaspati and Manuel Barriola
Tufts Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155
and Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
(Received 7 May 1992)

We find the conditions for the existence of a new class of defects. These defects arise from the nontrivial homotopy associated with the spontaneous breaking of subgroups of the full symmetry group of the field theory and may be viewed as topological defects embedded in a larger theory. Examples include monopole configurations in an O(4) model and a new vortex solution in the Weinberg-Salam model.

PACS numbers: 11.15.Ex, 11.17.+y

The discovery [1-3] of vortex solutions in the Weinberg-Salam model of the electroweak interactions raises the possibility that such solutions may exist in a wider class of field theories. In this Letter, we show that this is indeed the case and give general conditions under which such solutions will exist. We give two new examples of such defects: The first is a monopole solution in an O(4) model while the second example describes a new string solution in the Weinberg-Salam model.

It should be mentioned at the outset that this new class of defects is not guaranteed to be stable and the stability needs to be treated case by case. However, from the examples of the semilocal string [4-6] and the Z string [7] in the Weinberg-Salam model, we have learned that the stability of these string solutions is generally parameter dependent and, only after a careful analysis, can one decide when the defect is stable. The situation in the case of the monopole is clearer due to the very general analyses of Brandt and Neri [8] and Coleman [9]: The monopoles in the new class are always unstable [10,11].

We shall find the conditions for the existence of defects by considering the general energy functional for static field configurations,

$$E=\int d^3x\left[\tfrac{1}{4}G_{ij}^aG_{ij}^a+(D_j\phi)^\dagger(D_j\phi)+V(\phi)\right], \qquad (1)$$

where, $i,j=1,2,3$ and the group index, denoted by lower case latin indices, runs over the number of generators of the symmetry group. The field strengths and covariant derivatives are defined by

$$G_{ij}^a=\partial_iA_j^a-\partial_jA_i^a+gf^{abc}A_i^bA_j^c, \qquad (2)$$

$$D_i=\partial_i-i\tfrac{1}{2}g\mathbf{T}^aA_i^a, \qquad (3)$$

where $\mathbf{T}^a$ are the generators of the symmetry group G and g is the charge. For simplicity we have considered a compact group and, hence, only one charge in the model but it makes no difference to the arguments below if we have a direct product of groups with several different charges.

The potential for the Higgs field ϕ may be taken to be the usual Mexican hat potential. This results in the spontaneous breaking of the symmetry G once ϕ acquires a vacuum expectation value (VEV). The final symmetry group will be denoted by H.

We first wish to consider a nontrivial field configuration $\phi=\phi_0$, $A_i^a=B_i^a$ that describes a defect [12]. This means that the fields are in the configuration of a string [13] or a monopole [14]. These defects are composed of components of the Higgs field and some of the gauge fields. Therefore we adopt the notation that the Higgs field components and the gauge fields that constitute the string or monopole will be labeled by upper case latin and by greek indices, respectively. The other fields will be labeled by corresponding barred indices. That is, the defect configuration is $\phi^J=\phi_0^J$, $A_i^\beta=B_i^\beta$, $\phi^{\bar{J}}=0$, and $A_i^{\bar{\beta}}=0$. (Note that a complex scalar field is considered as having two components and so all the components of ϕ are real.)

Before proceeding further, for clarity, let us summarize our notation for the various indices. In the general analysis that follows, $i,j,\ldots$ denote spatial indices; $a,b,\ldots$ denote group indices and run over the entire number of group generators; $J,K,\ldots$ label those components of ϕ which are nontrivial in the defect configuration; $\alpha,\beta,\ldots$ label those gauge fields that are nontrivial in the defect; barred indices such as $\bar{J},\bar{K},\ldots$ and $\bar{\alpha},\bar{\beta},\ldots$ label the trivial (vanishing) components of ϕ and the gauge fields, respectively.

We now wish to find the conditions under which the defect configuration extremizes E. For this to happen, the linear order variation of E, when ϕ and A_i^a are perturbed around (ϕ_0,B_i^a) must vanish. Then, the basic idea is to rewrite the energy functional as

$$E=(E_{\text{defect}}+\delta E_{\text{defect}})+\delta E_{\text{other}} \qquad (4)$$

when the fields are perturbed around the defect configuration. The term E_{defect} represents the energy in the defect configuration and only depends on the fields that make up the defect. The variation in the energy functional E_{defect} is written as δE_{defect} and this only depends on the perturbations in the fields of the defect. The last term, δE_{other}, contains variations in the energy due to the perturbations in the "extra" fields—in the fields that do not constitute the defect.

We will assume for the time being that we have chosen the defect configuration suitably so that, if the model only consisted of the fields labeled by the unbarred indices,

© 1992 The American Physical Society

then the field configuration really does describe a defect. In other words, if the energy functional was simply given by $E_{\rm defect}$, then the configuration ϕ_0^J, B_i^α would minimize $E_{\rm defect}$. This can be guaranteed if the model given by $E_{\rm defect}$ alone has *topological* defects. Therefore, if the symmetry group of $E_{\rm defect}$, G_1, breaks down to H_1 once the ϕ^J acquire a VEV, then we need

$$\pi_n(G_1/H_1)\neq 1, \quad n=1 \text{ or } 2\,, \tag{5}$$

for there to be a defect solution in the truncated model given by $E_{\rm defect}$. This immediately implies that the variation $\delta E_{\rm defect}$ is quadratic in the perturbations. (It can further be assumed that the defect is stable in itself and therefore $\delta E_{\rm defect}$ is non-negative up to quadratic order in the perturbations. This is certainly true of Nielsen-Olesen strings with unit winding number and 't Hooft-Polyakov monopoles with unit magnetic charge.) The next task is to see when $\delta E_{\rm other}$ will be quadratic in the perturbations.

Consider the $G_{ij}^a G_{ij}^a$ term in E first. To decompose the energy functional in the form of (4), we write

$$G_{ij}^a G_{ij}^a=(\partial_i A_j^\alpha-\partial_j A_i^\alpha+gf^{\alpha bc}A_i^b A_j^c)^2 + (\partial_i A_j^{\bar\alpha}-\partial_j A_i^{\bar\alpha}+gf^{\bar\alpha bc}A_i^b A_j^c)^2\,. \tag{6}$$

(Note that lower case latin indices run over the entire number of generators.) For the right-hand side to be of quadratic order in the "other" perturbations, we should not have any terms that are linear in the gauge fields labeled by the barred greek indices. This can only be if

$$f^{\alpha\bar\beta\gamma}=0=f^{\alpha\beta\bar\gamma}=f^{\bar\alpha\beta\gamma}\,. \tag{7}$$

This condition implies that the algebra of the generators $\mathbf{T}^\alpha$ corresponding to the gauge fields making up the defect must close. That is, the algebra must be a subalgebra and must generate a subgroup of G. This subgroup is precisely the symmetry group of $E_{\rm defect}$ and has been previously denoted by G_1.

Next, we consider the term $|D_i\phi|^2$ in E. We write

$$|D_i\phi|^2=|d_i\phi|^2+J_i^{\bar\alpha}A_i^{\bar\alpha}+\cdots\,, \tag{8}$$

where the terms that have been omitted are already quadratic in the perturbations $A_i^{\bar\alpha}$ and

$$d_i=\partial_i-i\tfrac{1}{2}g\mathbf{T}^\alpha A_i^\alpha\,, \tag{9}$$

$$J_i^{\bar\alpha}=\tfrac{1}{2}g[\phi^\dagger\mathbf{T}^{\bar\alpha\dagger}d_i\phi-(d_i\phi)^\dagger\mathbf{T}^{\bar\alpha}\phi]\,. \tag{10}$$

Therefore, for $|D_i\phi|^2$ to be quadratic in the perturbations, we need $J_i^{\bar\alpha}$ for the defect configuration to be zero. The defect fields must satisfy their own equations of motion and so we cannot impose any additional requirements on their spatial dependence. A little algebra then yields the following two conditions:

$$\phi_0^\dagger\mathbf{T}^{\bar\alpha\dagger}\partial_i\phi_0-\partial_i\phi_0^\dagger\mathbf{T}^{\bar\alpha}\phi_0=0\,, \tag{11}$$

$$\phi_0^\dagger(\mathbf{T}^{\bar\alpha\dagger}\mathbf{T}^\alpha+\mathbf{T}^{\alpha\dagger}\mathbf{T}^{\bar\alpha})\phi_0=0\,. \tag{12}$$

In addition we must require that $|d_i\phi|^2$ in (8) be quadratic in the barred perturbations. [The unbarred perturbations only enter the $\delta E_{\rm defect}$ term in (4).] This can be achieved provided $i\mathbf{T}^\alpha\phi_0$ has vanishing components in the barred directions. That is,

$$[i\mathbf{T}^\alpha\phi_0]^{\bar J}=0\,. \tag{13}$$

The conditions (11)-(13) have a simple geometric interpretation. The condition (13) may be rewritten as

$$[e^{i\omega_\alpha\mathbf{T}^\alpha}\phi_0]^{\bar J}=0 \tag{14}$$

for arbitrary ω_α. In other words, ϕ_0 must lie on the orbit of the subgroup G_1. Now with (13) in (10), the conditions (11) and (12) can be rewritten as

$$[e^{i\omega_{\bar\alpha}\mathbf{T}^{\bar\alpha}}\phi_0]^J=\phi_0^J+O(\omega^2)\,. \tag{15}$$

Therefore, infinitesimal group elements not belonging to the subgroup G_1 should rotate ϕ_0 in directions orthogonal to the orbit of G_1 [15].

The potential for the Higgs field must also be quadratic in the perturbations. This is automatically true for the Mexican hat potential since that can be written as

$$V(\phi)=\lambda\left(\sum_J(\phi^J)^2+\sum_{\bar J}(\phi^{\bar J})^2-\frac{\eta^2}{2}\right)^2, \tag{16}$$

which does not contain any terms of linear order in $\phi^{\bar J}$. If the potential is not simply the Mexican hat potential but of some other form, the condition that the potential must satisfy may be written as

$$V_{,\bar J}(\phi_0)=0\,. \tag{17}$$

In the examples that follow we will only consider the Mexican hat potential for which (17) is always satisfied.

The conditions (5), (7), (11)-(13), and (17) are the conditions necessary for the field configuration ϕ_0, B_i^α to extremize the energy functional. The first two conditions are requirements on the group structure of the theory. The conditions (11)-(13) may be viewed as conditions on the field ϕ_0 whereas (17) is the condition on the potential. We now consider two examples.

The first model we consider is the $O(4)\to O(3)$ model in which the Higgs field is a four-vector. There are six generators of O(4) which we call τ^i,σ^i where $i=1,2,3$. The Lie algebra is

$$[\tau^i,\tau^j]=-i\epsilon^{ijk}\tau^k,\quad [\tau^i,\sigma^j]=-i\epsilon^{ijk}\sigma^k\,,$$
$$[\sigma^i,\sigma^j]=-i\epsilon^{ijk}\tau^k\,. \tag{18}$$

Explicitly, $(\tau^i)_{jk}=i\epsilon^{ijk}$ for $j,k=1,2,3$ and the other components of τ^i are zero. Also, $(\sigma^i)_{jk}=i(\delta_{ij}\delta_{k4}-\delta_{j4}\delta_{ik})$ for $j,k=1,2,3,4$.

The τ generators form a subalgebra and the corresponding subgroup is $O(3)_\tau$. We will choose ϕ_0 such that $O(3)_\tau$ is broken down to $O(2)_\tau$. Then the monopole solution is

$$\phi_0 = \begin{pmatrix} \vec{\phi}_{\text{tHP}} \\ 0 \end{pmatrix}, \quad A_i^a = [A_i^a]_{\text{tHP}}, \quad A_i^{\bar{a}} = 0\,, \tag{19}$$

where the subscript tHP stands to represent the 't Hooft-Polyakov solution, the index a runs from 1 to 3 and the index $\bar{a}$ runs from 4 to 6. The A_j^a fields are associated with the τ^i generators and the $A_j^{\bar{a}}$ fields with the σ^i. Now we can check that the conditions (5), (7), (11)-(13), and (17) are satisfied. (5) is satisfied because $\pi_2(\text{O}(3)/\text{O}(2)) = Z$. (7) is clearly satisfied by the Lie algebra in (12). (11) is satisfied because $\sigma^i\phi_0$ has a nonzero fourth component only and hence is orthogonal to ϕ_0. (12) and (13) are similarly true and (17) is automatically satisfied by the Mexican hat potential. Therefore (19) describes a monopole solution embedded in the O(4) model.

The stability of monopoles has been investigated in Refs. [8,9] under very general conditions. The analysis proceeds by first constructing (all) configurations that look like monopoles asymptotically in any non-Abelian theory with a compact group. Then the analysis shows that this asymptotic configuration is unstable to developing any other massless gauge fields that are present in the theory provided the monopole is not protected by topology. In our case, the residual symmetry is $O(3)$ and there are three massless gauge fields. One of the gauge fields is the radial magnetic field of the embedded monopole. Since the monopole is not topological, the monopole configuration is unstable towards driving the remaining two massless fields away from zero.

We should also mention that we can construct embedded *global* monopole solutions. The construction is identical to the O(4) monopole described above. Now, since there is no long-range magnetic field, the analyses of Refs. [8,9] do not apply. However, a detailed stability analysis shows that the embedded global monopole solution too is unstable [16].

We now turn to embedded string solutions. By arguments similar to those for the O(4) monopole, it is easy to construct string solutions in a model with O(N) symmetry. However, we will not treat these in any detail. Instead we consider the Weinberg-Salam model as our second example. The symmetry breaking is $\text{SU}(2) \times \text{U}(1)_Y \to \text{U}(1)$ and the Higgs field is in the fundamental representation. In this scheme, there is no SU(2) subgroup of the initial symmetry group which is broken down to U(1) and hence we do not expect isolated magnetic monopole solutions. (Monopoles connected by strings are still possible [3].) However, there are several U(1) subgroups that are broken down completely and so there should be a corresponding number of string solutions. One such solution is the Z string described in Refs. [1,2]. We now show that at least one more solution is present.

Consider the U(1) subgroup generated by the SU(2) generator τ^1. A vacuum expectation value of ϕ will break this U(1) subgroup completely. Since $\pi_1(\text{U}(1)/1) = Z$, this will give us τ^1 strings in the Weinberg-Salam model.

Let us find the τ^1 string by examining each of the conditions (5), (7), (11)-(13), and (17). The condition (5) is satisfied since any nonvanishing ϕ_0 breaks the U(1) completely and $\pi_1(\text{U}(1)/1) = Z$. The condition (7) is trivially satisfied since we are considering a U(1) subgroup which is Abelian. The condition (11) requires

$$\phi_0 = \begin{pmatrix} \phi_1 \\ i\phi_2 \end{pmatrix}, \tag{20}$$

where ϕ_1 and ϕ_2 are real fields [17]. Condition (12) is satisfied by the SU(2) generators since, for this group, the generators are Hermitian and $\tau^i\tau^j + \tau^j\tau^i = 0$. One can also check that $\phi_0^\dagger \tau^1 1 \phi_0 = 0$ provided ϕ_0 has the form in (20). [1 is the generator of $\text{U}(1)_Y$.] It is easy to see that (13) is satisfied since $i\tau^1\phi_0$ has the same form as ϕ_0 in (20). Finally, (17) is always satisfied by the Mexican hat potential.

So now we can write down the τ^1 string solution:

$$\begin{aligned} &\phi_0 = f_{\text{NO}}(r) \begin{pmatrix} \cos\theta \\ i\sin\theta \end{pmatrix}, \quad W_i^1 = [A_i]_{\text{NO}}\,, \\ &W_i^2 = 0 = W_i^3 = B_i\,, \end{aligned} \tag{21}$$

where we have used the notation of Ref. [18], the subscript NO stands for the Nielsen-Olesen solution, and (r,θ) are polar coordinates in the xy plane. (The string is taken to lie along the z axis.)

For comparison, we also give the previously found Z string [19]:

$$\begin{aligned} &\phi_0 = f_{\text{NO}}(r) \begin{pmatrix} 0 \\ \cos\theta + i\sin\theta \end{pmatrix}, \\ &Z_i = [A_i]_{\text{NO}}, \quad W_i^1 = 0 = W_i^2 = A_i\,. \end{aligned} \tag{22}$$

It is easy to see that the Z string is distinct from the τ^1 string since the gauge invariant hypercharge field strength, F_{Bij}, is nonzero in the Z string but zero in the τ^1 string. Hence, the two solutions described above are two *different* string solutions in the Weinberg-Salam model [20].

The stability of the Z string has been analyzed in detail in Ref. [7] and it was found that a region in parameter space exists where the strings are stable to small perturbations. The τ^1 string solution, however, is likely to be unstable for all values of the parameters. This conjecture is based on the observation that the τ^1 string is embedded entirely in the SU(2) group and the $\text{U}(1)_Y$ group is merely a spectator. This corresponds to the case of the Z string with $\sin^2\theta_W = 0$ for which the results of the stability analysis [7] always show instability.

At this point we would like to make a general remark regarding the stability of embedded string solutions. It seems to us that the stability of the string depends crucially on the presence of several different gauge coupling constants in the model. Then, different gauge fields in the model couple to the Higgs field with different strengths. Let us suppose that there is only one gauge field with the largest coupling constant (for example, the Z field in the Weinberg-Salam model). If this is the gauge field of the string, then the string is similar to the semilocal string. (In the limit that the coupling constant is infinitely larger than the other gauge coupling constants, the string is exactly the semilocal string.) The stability of the semilocal string then suggests that the embedded string can also be stable for some range of parameters.

To summarize, we have found the conditions necessary for the existence of a class of defects in field theories. This class of defects occurs due to the nontrivial topology associated with the spontaneous breaking of subgroups of the full symmetry group. In other words, these defects are ordinary topological defects that have been embedded in a bigger theory. Vortex solutions will be extremely common since one can always find U(1) [or O(2)] subgroups that are broken down completely; monopole solutions will also be fairly common. We have applied the general conditions to two specific examples: First we found monopole solutions embedded in an O(4) model and then we found a new string solution in the Weinberg-Salam model.

We are grateful to Ed Copeland, Tom Kibble, John Preskill, and Erick Weinberg for discussions. This work was supported in part by the National Science Foundation under Grant No. PHY89-04035.

[1] T. Vachaspati, Phys. Rev. Lett. **68**, 1977 (1992).
[2] T. Vachaspati, Nucl. Phys. B (to be published).
[3] Y. Nambu, Nucl. Phys. **B130**, 505 (1977); N. S. Manton, Phys. Rev. D **28**, 2019 (1983).
[4] T. Vachaspati and A. Achúcarro, Phys. Rev. D **44**, 3067 (1991).
[5] M. Hindmarsh, Phys. Rev. Lett. **68**, 1263 (1992).
[6] A. Achúcarro, K. Kuijken, L. Perivolaropoulos, and T. Vachaspati, Nucl. Phys. B (to be published).
[7] M. James, L. Perivolaropoulos, and T. Vachaspati, CfA report, 1992 (to be published).
[8] R. Brandt and F. Neri, Nucl. Phys. **B161**, 253 (1979).
[9] S. Coleman, in *The Unity of the Fundamental Interactions*, edited by A. Zichichi (Plenum, New York, 1983).
[10] We would like to thank Erick Weinberg for bringing the analyses in Refs. [8] and [9] to our attention.
[11] J. Preskill, Report No. CALT-68-1787, 1992 (unpublished).
[12] We will only consider gauged monopoles and strings explicitly in this Letter. However, identical considerations also apply to global defects, domain walls, and textures.
[13] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[14] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
[15] These conditions may have a simple mathematical interpretation. We suspect that they amount to requiring that the orbit of G_1 be a geodesic on the vacuum manifold: $V(\phi)=0$.
[16] M. Barriola (unpublished); L. Perivolaropoulos (unpublished).
[17] Equivalently, from Eq. (14), we know that $\phi_0=\exp(i\tau^1\theta)\Phi$ where Φ is some fixed doublet.
[18] J. C. Taylor, *Gauge Theories of Weak Interactions* (Cambridge Univ. Press, Cambridge, 1976).
[19] The gauge coupling in the Z string is $(g^2+g'^2)^{1/2}$ where g is the SU(2) charge and g' is the hypercharge whereas for the τ strings the gauge coupling is g. The Nielsen-Olesen functions in (21) and (22) depend on the gauge coupling.
[20] In a previous version of this paper, it was incorrectly stated that another distinct string—the τ^2 string—is present in the Weinberg-Salam model. The τ^1 and τ^2 strings are actually gauge equivalent when one includes $U(1)_Y$ gauge transformations.