Comment on "Classical and Quantum Superdiffusion in a Time-Dependent Random Potential"

Golubović, Feng, and Zeng [1] treat the problem of particle diffusion in random potentials with Gaussian space-time correlations, i.e., $[V(\mathbf{x},t)V(\mathbf{x}',t')]=\exp[-(\mathbf{x}-\mathbf{x}')^2/l^2-(t-t')^2/\tau^2]$. It is our claim that Eq. (8) of their paper (and subsequent results) is incorrect since it violates Liouville's theorem: The phase-space distribution must obey $dP/dt=0$, i.e., $\partial P/\partial t=0$ for $\partial P/\partial x_i, \partial P/\partial v_i=0$, since this is true for each realization. (Here x_i, v_i are Cartesian position and velocity coordinates.)

To derive the proper governing equation we begin with the usual Fokker-Planck equation [2] derived from the continuity of P:

$$\frac{\partial P}{\partial t}+v_i\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial v_i}\left\{\langle\Delta v_i\rangle P-\frac{1}{2}\frac{\partial}{\partial v_j}\langle\Delta v_i\Delta v_j\rangle P\right\}=0.$$

Liouville's theorem implies that $\langle\Delta v_i\rangle=\frac{1}{2}\partial\langle\Delta v_i\Delta v_j\rangle/\partial v_j$. For a potential $V(\mathbf{x},t)$,

$$M_{ij}\equiv\langle\Delta v_i\Delta v_j\rangle\equiv\lim_{T\to\infty}\frac{1}{T}\left[\int_0^T\frac{\partial V}{\partial x_i'}(\mathbf{x},t')dt'\int_0^T\frac{\partial V(\mathbf{x}'',t'')}{\partial x_j''}dt''\right]\equiv\int_\infty^\infty d\tau\left[\frac{\partial V}{\partial x_i'}(\mathbf{x}',0)\frac{\partial V}{\partial x_j'}(\mathbf{x}'-\mathbf{v}\tau,\tau)\right]$$

with the square brackets indicating an ensemble average over the random potential and $\mathbf{x}'=\mathbf{x}''+\mathbf{v}(t'-t'')$. For a specific potential such as the Gaussian random potential of Golubović, Feng, and Zeng [1] it is easy to carry out the ensemble averages to evaluate M_{ij}. Thus we express

$$V(\mathbf{x},t)=\int d^3\mathbf{k}\,d\omega\,V_{\mathbf{k},\omega}\exp[-i\omega t+i\mathbf{k}\cdot\mathbf{x}]$$

and take

$$[V_{\mathbf{k},\omega}V^*_{\mathbf{k}',\omega'}]=C\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\exp\left(-\frac{k^2}{k_0^2}-\frac{\omega^2}{\omega_0^2}\right)$$

after which the $t'-t''$, ω, and $\mathbf{k}$ integrals in M_{ij} are easily performed to yield, instead of the paper's Eq. (8), the correct Fokker-Planck equation,

$$\frac{\partial P}{\partial t}+v_i\frac{\partial P}{\partial x_i}-\frac{1}{2}\frac{\partial}{\partial v_i}M_{ij}\frac{\partial P}{\partial v_j}=0,\qquad(1)$$

with the velocity diffusion tensor

$$M_{ij}=\left\{\left(1+\frac{v^2}{v_0^2}\right)^{-3/2}\frac{v_iv_j}{v^2}+\left(1+\frac{v^2}{v_0^2}\right)^{-1/2}\left(\delta_{ij}-\frac{v_iv_j}{v^2}\right)\right\}$$

as correctly given by Eq. (6) of the paper [1]. Here $v_0=l/\tau$ with the diffusion coefficient decreasing for $v>v_0$ as the particle moves rapidly through the correlation length of the potential.

To solve the problem of diffusion from a point source at $\mathbf{x}=0$ it is useful to take spatial moments of Eq. (1). Define $P_0=\int P\,d^d\mathbf{x}$, $Q_k=\int Px_k\,d^d\mathbf{x}\equiv(v_k/v)P_1$, where P_0 and P_1 are isotropic functions of $\mathbf{v}$. Integrating Eq. (1) over all space gives, with d the number of dimensions,

$$2\frac{\partial P_0}{\partial t}=L_0P_0=v^{-(d-1)}\frac{\partial}{\partial v}\left(\frac{v^{d-1}}{(1+v^2/v_0^2)^{3/2}}\right)\frac{\partial P_0}{\partial v},\qquad(2)$$

while for the first "spherical" harmonic, which measures the correlation of the velocity direction with the direction from the origin of the motion, we have from multiplying Eq. (1) by x_k and integrating over all space

$$2\frac{\partial P_1}{\partial t}=L_0P_1-(d-1)\left[v^2\left(1+\frac{v^2}{v_0^2}\right)^{1/2}\right]^{-1}P_1+vP_0.\qquad(3)$$

The rate of change of $\langle x^2\rangle$ is given by the x^2 moment of Eq. (1).

For early times when $\langle v^2\rangle\ll v_0^2$ we approximate $(1+v^2/v_0^2)^{1/2}=1$ and recover the usual results $\langle v^2\rangle\sim t$, $\langle x^2\rangle\sim t^3$ for any dimension d.

For late times when we approximate $(1+v^2/v_0^2)^{1/2}=v/v_0$ it is clear from Eq. (2) that P_0 is of the form $t^{-d/5}f_0(v^5/t)$ so that $\langle v^2\rangle\sim t^{2/5}$ for all dimensions d.

Now, because the angular diffusion term [the second on the right-hand side of Eq. (3)] is large compared to the energy diffusion at large v, by a factor $v^2\sim t^{2/5}$, the particle changes direction more frequently than the velocity buildup, leading to smaller diffusion for $d>1$. Hence for $d=1$, $P_1\sim v^6P_0$ and for $d>1$, $P_1\sim v^4P_0$. Thus, $\langle x^2\rangle\sim t^{12/5}$ $(d=1)$ and $\langle x^2\rangle\sim t^2$ $(d>1)$. So we find different scalings for $\langle v^2\rangle$ and $\langle x^2\rangle$ in the case $d>1$ from those given in the paper [1]. Finally, since quantum spreading $\langle x^2\rangle\sim t$, it is obvious that quantum effects (for continuous potentials such as considered here) are only relevant at very early times and need not be considered. These comments do not alter the basic conclusions of the paper.

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[1] L. Golubović, S. Feng, and F.-A. Zeng, Phys. Rev. Lett. **67**, 2115 (1991).
[2] See, for example, S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943), or any standard text.

