# Autoresonant Three-Wave Interactions

L. Friedland
*Racah Institute of Physics, Hebrew University of Jerusalem, 91904 Jerusalem, Israel*
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The autoresonance in nonlinear three-wave interactions is discussed. The phenomenon is characteristic of inhomogeneous (or time-dependent) media in the presence of nonlinear wave-vector (frequency) shifts and leads to a persistent and efficient redistribution of the wave action fluxes (action densities) via a continuing spatial (temporal) self-adjustment of the nonlinear resonance relation. Conditions for the autoresonance are found and the phenomenon is illustrated by numerical examples.

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Nonlinear three-wave resonant interactions (3WRI) are of great importance, since they describe the lowest order (in terms of wave amplitudes) nonlinear effects in the evolution of perturbations in many physical systems [1]. Historically, the theory of the 3WRI proceeded from homogeneous and time-independent equilibria, through the inclusion of a one-dimensional inhomogeneity [2,3] and nonlinear frequency shifts [4], and, finally, to the three-dimensional evolution of the interaction in uniform [1] and one-dimensional [5] media.

A 3WRI process requires satisfaction of the resonance relations $\omega_1=\omega_2+\omega_3$ and $\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3$ for the frequencies and wave vectors of the interacting waves. Therefore, any factor affecting these relations makes a large influence on the 3WRI. Inhomogeneity, for example, limits the effective spatial three-wave interaction range as the wave vectors detune from the exact resonance relation. A classical application of this phenomenon is the stabilization of the explosive instability (one of the important cases of the 3WRI) by an inhomogeneity [2]. Nonlinear frequency shifts are also known to stabilize the explosive instability [4] because of the dependence of the resonance relations on the amplitudes of the waves. Our goal is to consider the *combined* effect of inhomogeneity (or time dependence) of the medium and nonlinearity of the wave dispersions on the 3WRI. We shall see that the width of the resonant interaction region in an inhomogeneous medium can be significantly *broadened* or even become infinite, with the addition of the nonlinearity, provided the system enters the *autoresonant* interaction regime (hereafter referred to as the 3WARI). The autoresonance phenomenon was studied previously in the context of accelerators [6], atomic physics [7], nonlinear dynamics [8], and pairwise mode conversion [9], and this Letter presents an extension of the idea to the 3WRI.

The starting point of our theory is the system of 3WRI equations in a one-dimensional, weakly inhomogeneous and slowly time-dependent medium,

$$L_1(D_1)A_1+i(D_1+\delta D_1)A_1=\varepsilon A_2A_3\exp(i\Psi)\,,$$
$$L_2(D_2)A_2+i(D_2+\delta D_2)A_2=-\varepsilon^*A_1A_3^*\exp(-i\Psi)\,, \quad (1)$$
$$L_3(D_3)A_3+i(D_3+\delta D_3)A_3=-\varepsilon^*A_1A_2^*\exp(-i\Psi)\,,$$

describing the evolution of the complex amplitudes $A_i$ ($i=1,2,3$) of three interacting waves $Z_i(x,t)=A_i(x,t)\times\exp[\Psi_i(x,t)]$ for which $D_i(k_i,\omega_i;x,t)$ are the *real* linear local dispersion functions, $\delta D_i$ are the lowest-order *real* nonlinear corrections to $D_i$, $k_i=\partial\Psi_i/\partial x$ and $\omega_i=-\partial\Psi_i/\partial t$ are the wave vectors and frequencies, $\varepsilon$ is the complex wave coupling coefficient, and $\Psi=\Psi_2+\Psi_3-\Psi_1$. The operators $L_i$ in (1) are defined via

$$L_i(D_i)=-(\partial D_i/\partial\omega_i)(\partial/\partial t)+(\partial D_i/\partial k_i)(\partial/\partial x)-\tfrac{1}{2}[d(\partial D_i/\partial\omega_i)/dt-d(\partial D_i/\partial k_i)/dx]\,.$$

In the case $\varepsilon=\delta D_i=0$, Eqs. (1) describe three *independent* geometric-optics modes [10] and the solution of the problem can be found by integrating along the rays for which $D_i=0$ serve as Hamiltonians. For a weak nonlinearity, we can still define $\omega_i$ and $k_i$ by the ray equations of the linear problem, i.e., set $D_i=0$ in (1) for nonzero $\varepsilon$ and $\delta D_i$.

At this stage, we reduce the complexity of the problem and consider a stationary, but weakly $x$-dependent background case (a uniform, but time-dependent situation can be treated similarly). Then $A_i=A_i(x)$, $\omega_i=\text{const}$, and $D_i(k_i,\omega_i;x)=0$ is viewed as the equation for $k_i=k_i(x)$. We also assume that the wave frequencies satisfy the exact resonance relation $\omega_1=\omega_2+\omega_3$ and, therefore, $\Psi=\Psi(x)$ and Eqs. (1) can be rewritten as

$$s_1dA_1'/dx+i\delta k_1A_1'=p_1\varepsilon'A_2'A_3'\exp(i\Psi)\,,$$
$$s_2dA_2'/dx+i\delta k_2A_2'=-p_2\varepsilon'^*A_1'A_3'^*\exp(-i\Psi)\,, \quad (2)$$
$$s_3dA_3'/dx+i\delta k_3A_3'=-p_3\varepsilon'^*A_1'A_2'^*\exp(-i\Psi)\,,$$

where $s_i$ and $p_i$ are the signs of the group velocities $v_i=-(\partial D_i/\partial k_i)/(\partial D_i/\partial\omega_i)$ and action densities $W_i=-(\partial D_i/\partial\omega_i)|A_i|^2$ of the three waves, respectively, $A_i'=|\partial D_i/\partial k_i|^{1/2}A_i$, $\varepsilon'=\varepsilon|(\partial D_1/\partial k_1)(\partial D_2/\partial k_2)(\partial D_3/\partial k_3)|^{-1/2}$, and the nonlinear wave vector shifts are defined via $\delta k_i=p_i\delta D_i|(\partial D_i/\partial k_i)|^{-1}$.

Now, for simplicity, let us further restrict the analysis to the situation when $s_i>0$ (the three waves propagate in the positive $x$ direction) and introduce the absolute values $B_i$ and complex phases $\phi_i$ of the wave amplitudes. Then (2) yields the following system of real equations:

$$dB_i/dx=p_i'\eta B_jB_k\sin\Phi,\quad i=1,2,3,\quad j\neq k\neq i\,,$$
$$d\Phi/dx=\kappa(x)-\delta k+\eta G\cos\Phi\,, \quad (3)$$

where $\Phi=\Psi-\phi_1+\phi_2+\phi_3+\theta+\pi/2$, $\kappa(x)=k_2(x)+k_3(x)-k_1(x)$, $\delta k=\delta k_2+\delta k_3-\delta k_1$, $p_1'=p_1$, $p_{2,3}'=-p_{2,3}$, and $G=p_\alpha' B_\beta B_\gamma/B_\alpha+p_\beta' B_\alpha B_\gamma/B_\beta+p_\gamma' B_\alpha B_\beta/B_\gamma$, while $\eta$ and $\theta$ are the absolute value and complex phase of $\varepsilon'$. In order to close the system, we must relate the nonlinear wave vector shift $\delta k$ to $B_i$. To the lowest order [11], $\delta k=\sum\beta_i B_i^2$, where the coefficients $\beta_i$ are constants for simplicity. Finally, Eqs. (3) yield the Manley-Rowe conditions $p_i' B_i^2-p_j' B_j^2=M_{ij}=\text{const}$.

The Manley-Rowe conditions allow us to express any pair of the amplitudes $B_i$ (say $B_\alpha$ and $B_\beta$) via the third amplitude (denoted by $B_\gamma$ below) and therefore (3) defines a two-degrees-of-freedom problem for $B_\gamma$ and $\Phi$:

$$dB_\gamma/dx=\eta F(B_\gamma)\sin\Phi\,, \tag{4}$$

$$d\Phi/dx=\kappa(x)-\alpha_\gamma-h_\gamma B_\gamma^2+\eta G(B_\gamma)\cos\Phi\,, \tag{5}$$

where $F=p_\gamma' B_\alpha B_\beta$, $\alpha_\gamma=B_\alpha M_{\alpha\gamma}/p_\alpha'+\beta_\beta M_{\beta\gamma}/p_\beta'$, and $h_\gamma=\beta_\alpha p_\gamma'/p_\alpha'+\beta_\beta p_\gamma'/p_\beta'+\beta_\gamma$. Equations (4) and (5) yield, under certain conditions, the *autoresonant* regime in which the *nonlinear resonance* relation

$$\kappa(x)-\delta k(B_\gamma)=\kappa(x)-\alpha_\gamma-h_\gamma B_\gamma^2\approx 0 \tag{6}$$

is satisfied continuously, despite the spatial variation of the wave-vector shift $\kappa$. The departure in (6) from zero is of an oscillating nature (see below) so that the exact resonance relation is satisfied only in average.

Now let us find the conditions for the autoresonance. One of the conditions is the smallness of the coupling parameter $\eta$ (the dimensionless smallness criterion will be given below). We seek solutions for $B_\gamma$ oscillating around a slowly varying average $B_{\gamma 0}(x)$ and assume that the amplitude of these oscillations is small [of $O(\eta^{1/2})$]. We also assume that $B_{\gamma 0}$ satisfies the exact nonlinear resonance condition $\kappa-\alpha_\gamma-h_\gamma B_{\gamma 0}^2=0$ at the initial point of integration and start our analysis from the case when all $B_i$ are comparable (the important situation when $B_\gamma\ll B_{\alpha,\beta}$ and $\kappa-\alpha_\gamma-h_\gamma B_{\gamma 0}^2\neq 0$ on the boundary will be discussed later). In this case the $O(\eta)$ term with $\cos\Phi$ in (5) can be neglected yielding $d\Phi/dx=\kappa(x)-\alpha_\gamma-h_\gamma B_\gamma^2$. By differentiating this equation and using (4), we obtain

$$d^2\Phi/dx^2=d\kappa/dx-Sv_0^2\sin\Phi\,, \tag{7}$$

where $S=p_\gamma'\,\mathrm{sgn}(h_\gamma)$ and, to lowest order, $v_0^2=2\eta|h_\gamma|B_{\alpha 0}\times B_{\beta 0}B_{\gamma 0}$. Equation (7) shows that $\Phi$ can be viewed as the angle variable of an adiabatic nonlinear pendulum under the action of normalized "torque" $d\kappa/dx$. Then, if $d\kappa/dx$ is sufficiently small and, initially, $(d\Phi/dx)^2<2v_0^2(1+S\cos\Phi)$, i.e., one starts on a trapped trajectory in the $(\Phi,d\Phi/dx)$ phase plane of the pendulum, the angle $\Phi$ will perform nonlinear oscillations around the equilibrium point $\Phi_0$ which is near either 0 or $\pi$ for $S$ positive or negative, respectively. In order to describe the phenomenon in detail, we write the solution in the form $B_\gamma=B_{\gamma 0}+b_\gamma$ and $\Phi=\Phi_0+\phi$, where, as stated, the oscillating component $b_\gamma$ is of $O(\eta^{1/2})$. The smallness of $\phi$ is not required, but $\Phi_0$ is assumed to be near 0 or $\pi$ depending on $S$. Then, by separating the slowly varying and rapidly oscillating parts in (4) and (7), we have

$$dB_{\gamma 0}/dx=\eta F_0\sin\Phi_0\langle\cos\phi\rangle\,, \tag{8}$$

$$d\kappa/dx-Sv_0^2\sin\Phi_0\langle\cos\phi\rangle=0\,, \tag{9}$$

$$db_\gamma/dx=S\eta F_0\sin\phi\,, \tag{10}$$

$$d^2\phi/dx^2=-v_0^2\sin\phi\,, \tag{11}$$

where $F_0=F(B_{\gamma 0})$ and $\langle\cdots\rangle$ means averaging over the fast oscillations. Equation (11) is the adiabatic pendulum equation for the oscillating component $\phi$ of the phase mismatch. The knowledge of $\phi$ allows us to find $b_\gamma$ from Eq. (10). For example, Eq. (10) yields the condition

$$\eta F_0/v_0 B_{\gamma 0}=(\eta B_{\alpha 0}B_{\beta 0}/2|h_\gamma|B_{\gamma 0}^3)^{1/2}\ll 1 \tag{12}$$

guaranteeing the assumed relative smallness of $b_\gamma$. This inequality can also be viewed as the desired dimensionless criterion on the smallness of the coupling parameter $\eta$. The solution of (11) can be employed to evaluate $\langle\cos\phi\rangle$ for the use in Eqs. (8) and (9) for the averaged quantities. For example, Eq. (9) yields $\sin\Phi_0=(d\kappa/dx)v^{-2}\times\langle\cos\phi\rangle^{-1}$ which, by recalling the assumption of $\Phi_0$ being near 0 or $\pi$, leads to the adiabaticity criterion

$$|d\kappa/dx|\ll v_0^2\langle\cos\phi\rangle\,. \tag{13}$$

Finally, the substitution of the expression for $\sin\Phi_0$ into (8) yields $d[\kappa-h_\gamma B_{\gamma 0}^2]/dx=0$, proving the autoresonance relation $\kappa(x)-\alpha_\gamma-h_\gamma B_{\gamma 0}^2=0$ for the averaged quantities, since this relation is satisfied at the boundary. In conclusion, when all $B_i$ are comparable, the sufficient autoresonance conditions are (a) starting in resonance at the boundary and (b) satisfaction of (12) and (13).

Now we illustrate our theory by solving Eqs. (3) numerically. Figure 1 shows the results of the calculations for $\Phi$ (dashed line) and normalized wave action fluxes $N_i=B_i^2/B_1^2(x_0)$ (solid lines), $x_0=0$ being the initial integration point. We assumed a linear dependence $\kappa(x)=cx$ ($c>0$) and used the dimensionless coordinate $s=x\sqrt{c}$ in Fig. 1, for convenience. The following set of dimensionless parameters and initial values was employed in the calculations: $\eta B_1(x_0)/\sqrt{c}=0.3$, $\beta_1 B_1^2(x_0)/\sqrt{c}=20$, $\beta_2 B_1^2(x_0)/\sqrt{c}=-10$, $\beta_3=0$, $p_1=p_2=p_3=1$, $N_2(x_0)=2$, $N_3(x_0)=1.5$, and $\Phi(x_0)=0.2\pi$. Note that the exact resonance $\kappa(s)-\delta k=0$ is arranged at $s=0$. We see in the figure that $\Phi$ is trapped in almost all the domain of the integration and oscillates, as expected, around the equilibrium value $\Phi_0\approx 0$. For trapped $\Phi$ in Fig. 1, $N_i$ oscillates around *linearly* varying values

$$N_{i0}=[\kappa(x)-\alpha_i]/h_i B_1^2(x_0)\,, \tag{14}$$

where the $h_i$ ($i=1,2,3$) differ by a factor $\pm 1$, as follows from the Manley-Rowe conditions. We observe in the figure that the frequency and the amplitude of the oscillations evolve adiabatically until, at $s\approx 47$, $\Phi$ escapes from the trapped region and $\langle N_i\rangle$ remain constant for $s>47$. The reason for the exit from the autoresonance is that

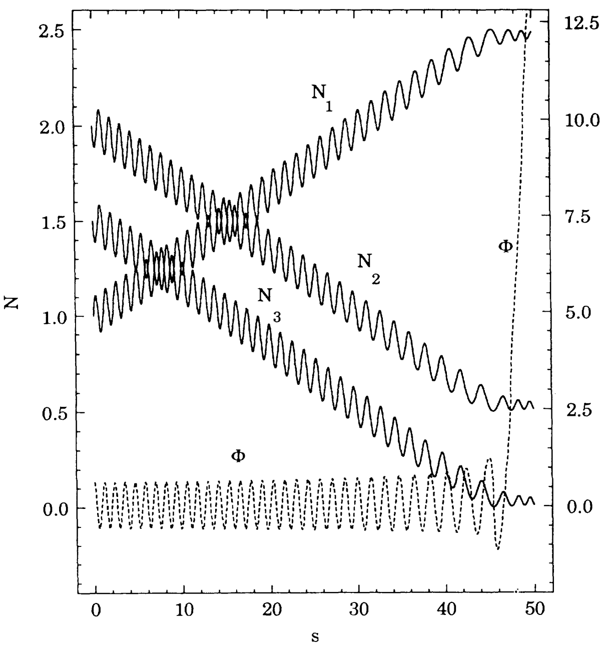


FIG. 1. The spatial evolution of Φ (dashed line) and $N_i$ (solid lines) for the resonance satisfied at the boundary.

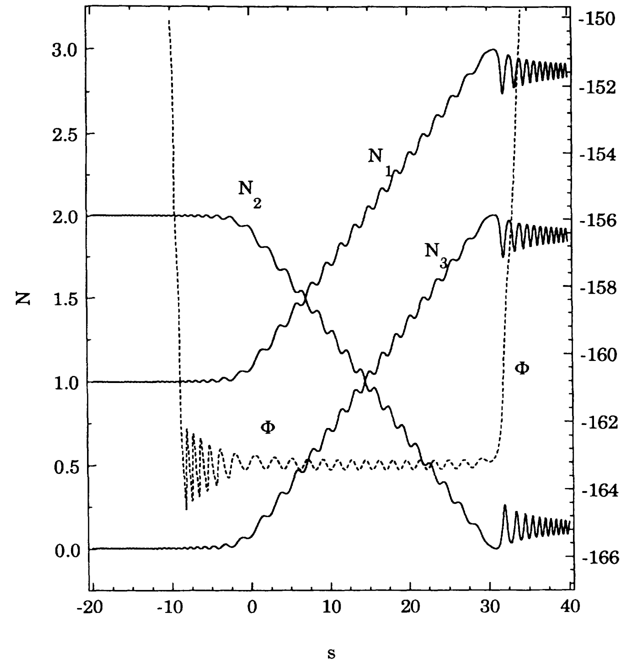


FIG. 2. The evolution of Φ (dashed line) and $N_i$ (solid lines) in the bounded 3WARI case with the resonance condition satisfied at the internal point.

one of the amplitudes ($B_3$) becomes small near the detrapping point violating (13).

The next important problem is associated with a more general situation in inhomogeneous media, when the resonance condition is *not* satisfied at the boundary, but, as $\kappa(x)$ varies in space, $\kappa-\alpha_\gamma-h_\gamma B\gamma^2$ passes through zero at some *internal* point $x_r$. Generally, in this case, one does not enter the autoresonant regime since the trapping of Φ must take place first. The trapping into the resonance, however, is a nonadiabatic phenomenon and thus, generally, is not characteristic of our slowly varying system. Nevertheless, there exists an important case, allowing the trapping despite the *adiabaticity* of the medium. Indeed, when one of the amplitudes (say $B_\gamma$) is sufficiently small, the second equation in (3) becomes singular due to the division by $B_\gamma$ in the term with cosΦ. Our recent analysis [9] of this singular situation in the context of the pairwise mode conversion showed that the trapping in this case is *guaranteed* if, far from $x_r$, $B_\gamma$ is sufficiently small and $\mathrm{sgn}(d\kappa/dx)=\mathrm{sgn}(h_\gamma)=\mathrm{sgn}(p'_\gamma)$. The arguments, similar to those of Ref. [9] with respect to the trapping, are applicable to the 3WRI case and will not be repeated here. The autoresonant interaction sets in after the trapping and continues as long as (12) and (13) are satisfied.

We shall conclude this Letter by discussing and illustrating the consequences of the autoresonance stage of the three-wave interaction in the above-mentioned case when the waves are not in the resonance at the boundary, but one of the waves is only weakly excited. The dependence of $N_{\gamma 0}=\langle N_\gamma\rangle$ on $x$ for the initially weakly excited wave has the simple form (14) in the autoresonant phase of the interaction, i.e., for $x>x_r$, where $x_r$ satisfies $\kappa(x_r)=\alpha_\gamma$. Equation (14) yields an increasing solution for $N_{\gamma 0}$ when $x>x_r$ since $\mathrm{sgn}(d\kappa/dx)=\mathrm{sgn}(h_\gamma)$. The average values of $N_\alpha$ and $N_\beta$ for the remaining two waves can also be found from (14) or directly from the Manley-Rowe conditions. Generally, the following two qualitatively different situations can be encountered in the process of the 3WARI in this case, depending on the energy signs $p_i$ of the interacting waves. The first can be referred to as the *bounded* 3WARI in which, due to the Manley-Rowe conditions, at least one of the remaining amplitudes *decreases* with the increase of $B_\gamma$. When the initially smallest *decreasing* amplitude (say $B_\alpha$) reaches the point where it becomes sufficiently small, the phase detrapping takes place, and the interaction results in a complete action flux exchange between modes $\alpha$ and $\gamma$. We illustrate the bounded 3WARI in Fig. 2, where the notations are similar to those in Fig. 1. We assumed the linear dependence $\kappa=cx$ ($c>0$), started the integration at $s=s_0=-20$, and used the following set of parameters and initial values: $\eta B_1(s_0)/\sqrt{c}=0.3$, $\beta_1 B_1^2(s_0)/\sqrt{c}=10$, $\beta_2 B_1^2(s_0)/\sqrt{c}=-5$, $\beta_3=0$, $p_1=p_2=1$, $p_3=-1$, $N_2(s_0)=2$, $N_3(s_0)=0.005$, and $\Phi(s_0)=0$. The figure shows the phase trapping and detrapping near the points $s=-9$ and 31, respectively. The characteristic phase oscillations around $\Phi_0(\mathrm{mod}2\pi)\approx 0$ and the linear dependence of $\langle N_i\rangle$ in the autoresonance can be observed. It is interesting to compare the results in Fig. 2 with those obtained by neglecting the nonlinear wave-vector shifts. This case is shown in Fig. 3 for the same example as in

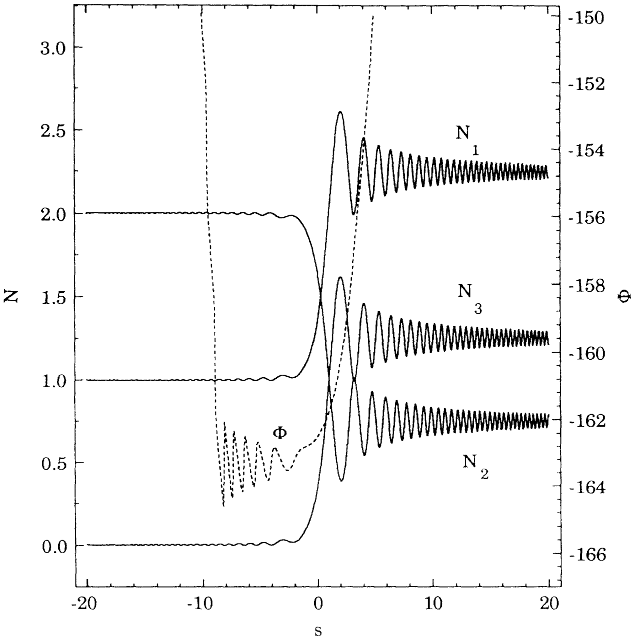


FIG. 3. The evolution of $\Phi$ (dashed line) and $N_i$ (solid lines) when the nonlinear wave vector shifts are neglected. The parameters and initial values are the same as in Fig. 2, but all $\beta_i=0$.

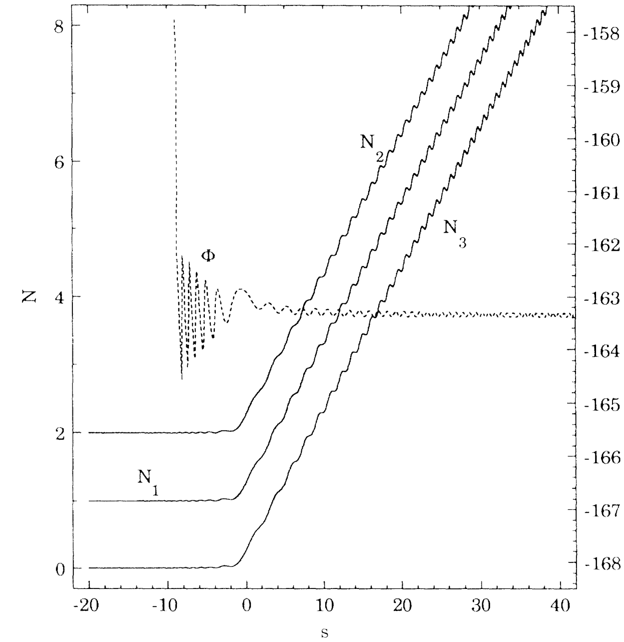


FIG. 4. The unbounded 3WARI. The parameters and the initial values are as those in Fig. 2, but $p_2=-1$.

Fig. 2, but with $\beta_1=\beta_2=0$. We see in the figure that the trapping stage proceeds similarly to the nonzero-$\beta_i$ case, but the detrapping takes place in the vicinity of the linear resonance point ($s=0$), destroying the interaction much earlier than in the 3WARI case (Fig. 2).

In contrast to the bounded 3WARI, the *unbounded* 3WARI case corresponds to the situation when, for growing $B_\gamma$ (the weakly excited mode), the Manley-Rowe conditions yield *growing* solutions for *both* the remaining waves. Such an example is shown in Fig. 4, where all the parameters and initial data are as in Fig. 2, but $p_2=-1$. After the trapping followed by the autoresonance in this case, one never obtains singular denominators in the second equation in (3) and the autoresonant spatial growth of the amplitudes continues indefinitely. In a uniform medium this situation corresponds to the explosive instability [4]. Inhomogeneity or nonlinearity separately are known to stabilize the instability in this case. The combination of both, however, may still result in an unbounded spatial growth of the wave amplitudes, but the instability loses its explosive character (asymptotically $B_i \sim |\kappa|^{1/2}$) and, thus, can be controlled by varying the parameters of the medium. The effect is similar to that in a tapered free-electron laser which can be viewed as a *degenerate* 3WRI with one of the waves (the undulator field) prescribed externally. The tapering of the undulator allows the nonlinear saturation of the laser signal to be avoided [12] and improves the laser efficiency.

In summary, we have presented a one-dimensional theory and examples of a new type of autoresonant three-wave interactions. The phenomenon requires the presence of the inhomogeneity (or time dependence) and nonlinear wave vector (frequency) shifts in the system and modifies, under specified conditions, both the character and overall efficiency of the interaction.

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