PHYSICAL REVIEW LETTERS

VOLUME 69 | 21 SEPTEMBER 1992 | NUMBER 12

Kinks and Topology Change

G. W. Gibbons and S. W. Hawking

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom

(Received 25 February 1992; revised manuscript received 10 July 1992)

We show that if a change of spatial topology is mediated by a spacetime with an everywhere-nonsingular metric of Lorentzian signature which admits a spinor structure, then the Kervaire semi-characteristic of the boundary plus the kink number of the Lorentzian metric on the boundary must vanish modulo 2. The kink number is a measure of how many times the light cone tips over on the boundary. It vanishes if the boundary is everywhere spacelike. This result gives a generalization of a previous selection rule: The number of wormholes plus the number of kinks created during a topology change is conserved modulo 2.

PACS numbers: 04.20.Cv, 02.40.+m

There has recently been a certain amount of discussion of two closely related ideas [1-3]. Both are matters of principle rather than practice. One is that sufficiently advanced civilizations might be able to change the topology of space by attaching handles, sometimes called wormholes, to spacetime. The other is that advanced civilizations might use these wormholes to build time machines. The issue is whether there is anything in the laws of physics to forbid such happenings. If the laws of physics which constrain all civilizations require an everywhere-nonsingular Lorentzian metric g_L and the existence of an $SL(2,C)$ spinor structure, one can make some general statements about topology-changing processes, using only kinematical information, independent of particular models or particular field equations. Some statements have been known for a long time. For example there certainly exist topology-changing Lorentzian spacetime manifolds but they suffer from the existence of closed timelike curves or are not time orientable [4-7]. It has also been known for some time that an arbitrary Lorentzian spacetime need not admit $SL(2,C)$ spinors [8,9]. The link between these two facts has been until recently only partially understood and perhaps not widely appreciated. With the renewed interest in wormholes (both the Lorentzian ones dealt with in this paper and the Euclidean, or more strictly, Riemannian, ones which have been invoked in attempts to solve the cosmological constant problem [10]), it seems worthwhile reexamining what connections there might be between causality and topology. In particular in a recent paper [11] attention was drawn to a selection rule governing changes of topology in *all* theories in which the spacetime manifold M has an everywhere-nonsingular Lorentz metric g_L. In [11] it was assumed that (i) M is compact, (ii) (M,g_L) is orientable and time orientable, and (iii) ∂M is spacelike.

One of the purposes of the present Letter is to relax assumption (iii) and consider boundaries ∂M which are partly spacelike and partly timelike. Each connected component Σ_a of the boundary ∂M is a closed connected orientable 3-manifold and we may associate with each such Σ_a an integer-valued invariant, $\text{kink}(\Sigma_a,g_L)$, which measures, roughly speaking, how many times the light cone of the metric g_L tips over on Σ_a. This invariant, originally introduced by Finkelstein and Misner [12,13], is called the kink number of the boundary component Σ_a. More precisely it may be defined as follows. Endow M with an auxiliary Riemannian metric g_R. Diagonalize the Lorentz metric g_L with respect to g_R at each point of M. The eigenvector with negative eigenvalue defines a line field $(\mathbf{V}, -\mathbf{V})$. If g_L is time orientable the ± 1 ambiguity may be resolved and $\mathbf{V}$ normalized with respect to the Riemannian metric g_R to give an everywhere-nonvanishing unit vector field on M, i.e., a global section of the sphere-bundle $S(M)$ of unit 4-vectors over M. We may pull back the bundle $S(M)$ to any connected component of the boundary Σ_a to obtain a six-dimensional bundle $S(\Sigma_a)$ with fiber, the 3-sphere. The bundle $S(\Sigma_a)$ has two global sections—that provided by the unit vector field $\mathbf{V}$ and that provided by the unit inward pointing normal $\mathbf{n}$ to the boundary component Σ_a. As with $\mathbf{V}$ the normal $\mathbf{n}$ is normalized using the Riemannian metric g_R. These two sections are three-dimensional submanifolds of the 6-manifold $S(\Sigma_a)$ and will intersect generically in a certain number of isolated points p_i. These points may be assigned a sign, ± 1, as follows: $S(\Sigma_a)$ is orientable, as are the two global sections. If the orientation of $S(\Sigma_a)$ at the point p_i coincides with the product of the orientations of the two global sections we assign the point p_i the value $+1$. We assign it the value -1 otherwise. The kink number, $\mathrm{kink}(\Sigma_a, g_L)$, is now defined to be the sum of the number of points p_i counted with regard to this sign.

Finkelstein and Misner [12] gave a different, but equivalent, definition of the kink number. In the present context, which is slightly different from theirs, it amounts to finding a framing of a collared neighborhood of the boundary component Σ_a of the form $(\mathbf{n}, \mathbf{e}_i)$ where $\{\mathbf{e}_i\}$, $i=1,2,3$, is some framing of Σ_a (since Σ_a is three dimensional such framings always exist). The components V^a of the vector field $\mathbf{V}$,

$$\mathbf{V} = V^0\mathbf{n} + V^i\mathbf{e}_i\,, \tag{1}$$

with

$$(V^0)^2 + V^1 V^i = 1\,, \tag{2}$$

define a map $f : \Sigma \to S^3$, the unit 3-sphere, and the kink number is defined to be the degree of this map. We may count the degrees by looking at the number of inverse images of a point $q \in S^3$. Choosing for q the point $(1, \mathbf{0})$ we see that the definition given by Finkelstein and Misner and that given above agree. Moreover the kink number is thus seen to be independent of the choice of framing $\{\mathbf{e}_i\}$. Since the space of Riemannian metrics on M is topologically trivial it is also independent of the choice of auxiliary Riemannian metric g_R and depends only on the Lorentzian metric g_L as our notation suggests.

We can extend the definition of kink number to any closed connected orientable three-dimensional hypersurface Σ lying in M, provided that Σ is provided with a direction for its unit normal $\mathbf{n}$. The kink number, $\mathrm{kink}(\Sigma, g_L)$, depends on this choice. Reversing the sign of the normal reverses the sign of the kink number.

If the boundary component Σ is everywhere spacelike or everywhere timelike then the kink number vanishes. An example of a boundary with nonvanishing kink number is obtained by considering a unit 4-ball B^4 in flat Minkowski spacetime. If $(t, \mathbf{x})$ are a set of inertial coordinates the ball is given by

$$t^2 + \mathbf{x}^2 \leq 1 \tag{3}$$

with boundary

$$t^2 + \mathbf{x}^2 = 1\,. \tag{4}$$

Choosing $\mathbf{V} = \partial/\partial t$ one sees that the kink number is one.

The definition we have given above was suggested to us by Graeme Segal. Now the following generalization of Hopf's theorem about nonvanishing vector fields holds [14]:

$$\chi(M) = \mathrm{kink}(\partial M, \mathbf{V})\,, \tag{5}$$

where the left-hand side of (5) is the sum of the kink numbers for each component.

Combining Eq. (5) with the result, proved in [11], that for M a spin-manifold

$$\chi(M) = \dim_{Z_2}[H_0(\partial M; Z_2) \oplus H_1(\partial M; Z_2)]\ (\mathrm{mod}\,2)\,, \tag{6}$$

we obtain the generalization of the original selection rule: For any 4-manifold M admitting $\mathrm{SL}(2,C)$ spinor structure,

$$\dim_{Z_2}[H_0(\partial M; Z_2) + H_1(\partial M; Z_2)] = \mathrm{kink}(\partial M, g_L)\ (\mathrm{mod}\,2)\,. \tag{7}$$

Thus, for example, if $\partial M = S^3 \sqcup S^1 \times S^2$, where $\sqcup$ denotes disjoint union, the left-hand side of (7) equals 1. This may be interpreted as saying that one cannot create a single wormhole in an S^3 universe if both components are spacelike. However, one may create a wormhole if one creates a kink as well. It is the number of wormholes plus the number of kinks that is conserved modulo 2 in this more general situation.

A simple example, but where neither the S^3 nor the $S^1 \times S^2$ components is spacelike, is provided by considering points in flat Minkowski spacetime inside the unit 4-ball given by Eq. (3) which are also outside an $S^2 \times B^2$ lying inside B^4 and given by, say,

$$(|\mathbf{x}| - \tfrac{1}{2})^2 + t^2 \geq \tfrac{1}{8}\,. \tag{8}$$

The boundary of M consists of the S^3 satisfying (4) and the $S^1 \times S^2$ for which

$$(|\mathbf{x}| - \tfrac{1}{2})^2 + t^2 = \tfrac{1}{8}\,. \tag{9}$$

One readily checks that M has Euler characteristic -1. The outer S^3 has kink number $+1$ and the inner $S^1 \times S^2$ has kink number -2. Since M is a subset of Minkowski spacetime it clearly admits $\mathrm{SL}(2,C)$ spinors. Alternatively consider $B^2 \times S^2$ with a spacelike boundary of topology $S^1 \times S^2$ described in [1]. Removing a small 4-ball creates another boundary component with topology S^3 and kink number -1. Thus a spacelike wormhole can disappear leaving an antikink in a universe of topology S^3.

Further simple examples may be obtained by removing from the unit 4-ball (3) n smaller, disjoint 4-balls. Each of the $(n-1)$ 3-spherical boundary components has kink number -1. We could also remove small 4-balls from the subset of Minkowski spacetime inside the $S^1 \times S^2$

given by Eq. (9), that is, for which the inequality is reversed in (8). This subset gives a compact spacetime with a wormhole boundary with kink number +2. The topology is $B^2 \times S^2$ which has Euler characteristic 2. Each small ball removed gives an S^3 boundary component with kink number -1 and the Euler characteristic decreases by unity.

One possible application of the kink concept, suggested to us by some remarks of John Barrett, might be to the construction of a topological field theory for four-dimensional Lorentzian spacetimes. A possible topological action would be some multiple α of the Euler number χ of the spacetime. If one takes the state space associated to each boundary component to be one-dimensional, the amplitude to go from one connected 3-manifold Σ_i with kink number k_i to another connected 3-manifold Σ_f with kink number k_f would be, by Eq. (7), $\exp[i\alpha(k_f - k_i)]$. We shall not explore this idea further here but pass on to considering the possible relation between kinks and causality.

If a boundary component is entirely spacelike then the kink number must vanish, though the converse is not necessarily true. If the spacetime has just one boundary component and if this boundary component is entirely spacelike then there must exist closed timelike curves in the interior. *This is the celebrated result of Geroch* [7]. An example of this was provided in [11] with a manifold topologically equivalent to $S^1 \times B^3$ with a spacelike boundary with wormhole topology $S^1 \times S^2$. One might be tempted to conjecture that any compact spacetime with a single boundary component with zero kink number must contain closed timelike curves. However, this is not true as is shown by the following example. Again we consider a compact subset of flat Minkowski spacetime with $S^1 \times S^2$ boundary. This time we consider the interior of a "solid torus," i.e., the set of points obtained from a unit 3-ball lying in a spacelike hyperplane of constant time and forming a solid revolution by rotating it in a timelike 2-plane along a circle of radius 4 say. If the initial 3-ball is given by

$$t=0, \quad x_1^2 + x_2^2 + (x_3 - 2)^2 \leq 1\,, \tag{10}$$

and we rotate in the x_3-t 2-plane, then the solid of revolution is obtained by replacing x_3 by $(x_3^2 + t^2)^{1/2}$ in Eq. (10). Clearing of surds we obtain

$$(\mathbf{x}^2 + t^2 + 3)^2 \leq 16(x_3^2 + t^2)\,. \tag{11}$$

Restricted to the initial hyperplane, Eq. (11) is satisfied by two solid 3-balls, the original one given by Eq. (10) and a second one, obtained by reflection in the x_3 direction, which is halfway around the circle. The subset of Minkowski spacetime defined by Eq. (11) has topology $S^1 \times B^3$ with a boundary, corresponding to equality in Eq. (11), with topology $S^1 \times S^2$. The flat Minkowski metric has zero kink number on this boundary and yet clearly there are no closed timelike curves.

Despite this last example one might have thought that there was some relation between kink number and causality. Consider for example any spacetime with a single boundary component topologically equivalent to the 3-sphere, S^3. We have seen in the examples above, and it follows quite generally, that removing small 4-balls creates boundary components with kink number -1. Moreover if the boundary is homotopically trivial it must clearly also have kink number $+1$. One is therefore tempted to conjecture that any compact spacetime M with boundary S^3 on which the kink number vanishes must contain closed timelike curves. However, in some recent unpublished work, done in response to an earlier version of this paper, A. Chamblin and R. Penrose have shown that for any boundary with any collection of kinks on it there exists a spacetime without closed timelike curves. Thus it seems that one cannot use the kink number as a diagnostic for closed timelike curves.

We would like to thank Roger Penrose and Graeme Segal for helpful conversations, especially the latter for help with Eq. (5). Andrew Chamblin has also obtained Eq. (7) and a number of other related results. We thank him for communicating these results to us.

[1] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446–1449 (1988).
[2] I. D. Novikov, Zh. Eksp. Teor. Fiz. **95**, 769 (1989) [Sov. Phys. JETP **68**, 439 (1989)].
[3] V. P. Frolov and I. G. Novikov, Phys. Rev. D **42**, 1057–1065 (1990).
[4] B. L. Rheinhardt, Topology **2**, 173–177 (1963).
[5] P. Yodzis, Commun. Math. Phys. **26**, 39–52 (1972).
[6] P. Yodzis, Gen. Relativ. Gravitation **4**, 299–305 (1973).
[7] R. P. Geroch, J. Math. Phys. **8**, 782–786 (1968).
[8] K. Bichteler, J. Math. Phys. **6**, 813–815 (1968).
[9] R. P. Geroch, J. Math. Phys. **9**, 1739–1744 (1968); **11**, 343–347 (1970).
[10] S. Coleman, Nucl. Phys. **B310**, 643–668 (1988).
[11] G. W. Gibbons and S. W. Hawking, "Selection Rules for Topology Change," Commun. Math. Phys. (to be published).
[12] D. Finkelstein and C. W. Misner, Ann. Phys. (N.Y.) **6**, 230–243 (1959).
[13] D. Finkelstein and C. W. Misner, *Further Results in Topological Relativity*, Les Theories Relativistes de la Gravitation, Royaumont, edited by A. Lichnerowicz and M. A. Tonnelat (Editions du CNRS, Paris, 1962).
[14] V. I. Arnold, *Singularity Theory*, London Mathematical Society Lecture Notes No. 53 (Cambridge Univ. Press, Cambridge, 1981).