Gauge-Independent Analysis of Chern-Simons Theory with Matter Coupling

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We show that the Chern-Simons theory coupled to complex scalars can be consistently quantized in the Hamiltonian formalism without gauge constraints. A new structure of the anyon operator displaying fractional spin and statistics follows logically from our analysis without any *ad hoc* assumptions.

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The explicit construction of anyon operators exhibiting fractional spin and statistics in a canonical framework in (2+1)-dimensional quantum field theory has remained controversial and debatable. Work in this direction was pioneered by Semenoff [1], whose construction involved formal manipulations leading to some controversy and criticisms [2-5]. Ideas [6] akin to Semenoff's [1] have also been considered but subject to the same criticisms [2-5]. Besides the usual criticisms we emphasize that until now only gauge-fixed Hamiltonian methods have been employed to discuss fractional spin and statistics of gauge-dependent objects (the anyon operators). It is uncertain, therefore, whether the observed effects are physical or mere gauge artifacts. Indeed sometimes different results with different gauge fixing have been reported [7].

In this Letter we circumvent all these problems by showing, for the first time, that the (2+1)-dimensional Chern-Simons (CS) theory coupled to complex scalars can be consistently quantized in the canonical formalism without any gauge fixing. All the space-time symmetries of the theory are preserved and the full Poincaré algebra holds.

Our analysis naturally leads to the construction of gauge-independent multivalued operators which create the physical states of the theory with arbitrary spin. We associate these operators with the anyon operators of the model. The structure of the anyon operator is completely new and, being gauge independent, the observed effects are physical. The anyon operators obey graded commutation relations which are compatible with the usual spin statistics theorem valid for fermions and bosons. Finally we show that the effect of the anyonic operators is to eliminate the gauge interactions from the Hamiltonian. Formal manipulations are avoided at all stages of the computations.

The Lagrangian of our model is given by

$$\mathcal{L}=(D_\mu\phi)^*(D^\mu\phi)+\frac{\theta}{4\pi^2}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda\,,$$

where

$$D_\mu=\partial_\mu+iA_\mu\,,$$

with

$$\varepsilon^{012}=1,\quad g_{\mu\nu}=(+1,-1,-1)\,.$$

It is invariant (up to a total divergence) under the gauge transformations

$$\phi(x)\to e^{i\alpha(x)}\phi(x)\,,\qquad A_\mu(x)\to A_\mu(x)-\partial_\mu\alpha(x)\,. \tag{1}$$

The canonical momenta are

$$\Pi^0=\frac{\partial\mathcal{L}}{\partial\dot{A}_0}=0,\quad \Pi_i=\frac{\partial\mathcal{L}}{\partial\dot{A}^i}=\frac{\theta}{4\pi^2}\varepsilon_{ij}A^j\,,$$

$$\Pi=\frac{\partial\mathcal{L}}{\partial\dot\phi}=(D_0\phi)^*,\quad \Pi^*=\frac{\partial\mathcal{L}}{\partial\dot\phi^*}=D_0\phi\,,$$

so that, according to Dirac's [8] classification, the primary constraints are

$$P_0=\Pi_0\approx 0\,,\qquad P_i=\Pi_i-(\theta/4\pi^2)\varepsilon_{ij}A^j\approx 0\,, \tag{2}$$

and the symbol $\approx$ stands for weak equality. The canonical Hamiltonian density is obtained from the Lagrangian via the Legendre transformation,

$$\mathcal{H}_c=\sum_{\chi=\text{fields}}\Pi_\alpha\dot\chi_\alpha-\mathcal{L}$$
$$=\Pi^*\Pi-A_0J_0-(D_i\phi)^*(D^i\phi)-\frac{\theta}{2\pi^2}\varepsilon^{ij}A_0\partial_iA_j\,,$$

where

$$J_\mu=i[(D_\mu\phi)^*\phi-\phi^*(D_\mu\phi)] \tag{3}$$

is the conserved current.

The primary Hamiltonian is given by

$$H_P=\int d^2x(\mathcal{H}_c+U_0\Pi_0+U_iP_i)\,,$$

where U_0,U_i are arbitrary multipliers. Conserving the primary constraints with H_P and using the fundamental Poisson brackets (PB),

$$\{A_\mu(x),\Pi^\nu(y)\}=g_\mu^\nu\delta^{(2)}(x-y)\,,$$
$$\{\phi(x),\Pi(y)\}=\{\phi^*(x),\Pi^*(y)]=\delta^{(2)}(x-y)\,,$$

yields the secondary constraint,

$$S=J_0+(\theta/2\pi^2)\varepsilon^{ij}\partial_iA_j\approx 0\,.$$

No further constraints are generated via this iterative procedure. We find that P_0 is first-class while P_i and S are second-class constraints. It is, however, essential to extract the maximal set of first-class constraints [9]. The following combination of the second-class constraints,

$$P=\partial^i P_i+S=\partial^i\Pi_i+J_0+\frac{\theta}{4\pi^2}\varepsilon^{ij}\partial_i A_j\approx 0\,, \tag{4}$$

is first class. The maximal set of first-class constraints is thus given by P_0 and P while P_i are second class. This completes our classification of constraints.

We next compute the Dirac brackets (DB) among the fundamental variables, generically denoted by χ,

$$\{\chi(x),\chi(y)\}_{\rm DB}=\{\chi(x),\chi(y)\}_{\rm PB}-\int dz\,dz'\{\chi(x),P_i(z)\}_{\rm PB}P_{ij}^{-1}(z,z')\{P_j(z'),\chi(y)\}_{\rm PB}\,,$$

where

$$P_{ij}^{-1}(x,y)=(2\pi^2/\theta)\varepsilon_{ij}\delta(x-y)$$

is the inverse of the matrix of the PB's among P_i and P_j. The DB's which differ from their PB's are

$$\{A_i(x),A_j(y)\}_{\rm DB}=\left(\frac{4\pi^2}{\theta}\right)^2\{\Pi_i(x),\Pi_j(y)\}_{\rm DB}=\frac{2\pi^2}{\theta}\varepsilon_{ij}\delta(x-y)\,, \tag{5}$$

$$\{A_i(x),\Pi_j(y)\}_{\rm DB}=\frac{g_{ij}}{2}\delta(x-y)\,,$$

which are compatible with setting the second-class constraint P_i [Eq. (2)] strongly zero.

The total Hamiltonian is given by

$$\mathcal{H}_T=\mathcal{H}_c+U\Pi_0+VP\,,$$

where U,V are arbitrary multipliers reflecting the gauge invariances in the theory associated with the first-class constraints. It is possible to choose two gauge constraints so that this arbitrariness is eliminated. This is the usual course followed in the literature [1,6,7].

An alternative approach [10], however, is to determine U and V so that the correct Heisenberg equations of motion,

$$\left\{\chi,\int\mathcal{H}_T\right\}_{\rm DB}=\partial_0\chi$$

calculated via the DB (5), are obtained. In the present case it is simple to check that U and V are given by the unique choice

$$U=\partial_0 A_0,\quad V=0\,.$$

The same analysis is next repeated for the momentum operator M_i defined via the canonical energy-momentum tensor

$$M_i^C=\theta_{0i}^C\,,$$

where

$$\theta_{\mu\nu}^C=\sum_{\varphi=\phi,\phi^*,A_\mu}\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi)}\partial_\nu\varphi-\mathcal{L}g_{\mu\nu}=(D_\mu\phi)^*(\partial_\nu\phi)+(D_\mu\phi)(\partial_\nu\phi^*)+\frac{\theta}{4\pi^2}\varepsilon_{\sigma\mu\lambda}A^\sigma\partial_\nu A^\lambda-\mathcal{L}g_{\mu\nu}\,.$$

The final expressions for the generators of space-time translations may be written in a Lorentz covariant form,

$$\theta_{0\mu}^T=\theta_{0\mu}^C+U_{0\mu}\Pi_0+V_{0\mu}P \tag{6}$$

with

$$U_{0\mu}=\partial_\mu A_0,\quad V_{0\mu}=0$$

so that

$$\left\{\chi,\int\theta_{0\mu}^T\right\}_{\rm DB}=\partial_\mu\chi\,.$$

It is simple to check that the constraints have vanishing DB with $\int\theta_{00}^T$ so that they are fixed in time. In the quantized version, therefore, $\theta_{0\mu}^T$ replaces $\theta_{0\mu}^C$ while the DB are transformed into commutators following the usual prescription $i\{,\}_{\rm DB}\to[,]$, and operator symmetrization is implied whenever products of operators occur. The other space-time generators (i.e., rotations and boosts) can be treated in an identical fashion. It is found that the fields transform normally and there are no anomalies. Finally it can be shown that the Poincaré algebra is satisfied on the physical states, which are annihilated by the first-class constraints. This completes our discussion of the quantization of the model without gauge constraints.

Next we try to obtain the operators which create the physical one-particle states of the model. The physical states are gauge invariant, and we can be assured of this property if the operators creating these states from the vacuum are also gauge invariant. Moreover, following usual convention [5] we define the one-particle states to be those states which carry one unit of the charge $Q=\int d^2x\,J_0$ [Eq. (3)], i.e., states

$$|1\rangle=\hat\phi|0\rangle$$

such that the creation operator $\hat\phi$ obeys

$$[J_0(x),\hat\phi(y)]=\delta^{(2)}(x-y)\hat\phi(y)\,. \tag{7}$$

It is easy to show that, because of the nontrivial commutator

$$[J_0(x),\phi(y)]=\delta^{(2)}(x-y)\phi(y) \tag{8}$$

following from the DB (5), a general structure for a gauge-invariant $\hat\phi$ satisfying (7) may be written,

$$\hat\phi(x)=\exp\left[\int dy\,\Omega(x-y)J_0(y)+i\int_{x_0}^x dy_i\,A_i(y)\right]\phi(x)\,, \tag{9}$$

where Ω is, as yet, an undetermined function and x_0 is some reference point. The line integration in (9) is performed along the straight path,

$$y_i=(x_0)_i+(x-x_0)_i t, \quad 0\le t\le 1\,. \tag{10}$$

For subsequent calculations we point out that

$$\int_{x_0}^{x} dy_i \int_{x_0}^{z} dz_j [A_i(y),A_j(z)]=0$$

which follows from the symmetry properties of the commutator (5) and the integration path (10).

In order to determine the function $\Omega(x-y)$ in (9), we first compute the general n-particle state functional,

$$|\psi_n\rangle=\left[\prod_{i=1}^{n}\hat\phi(x_i)\right]|0\rangle\,.$$

To simplify this, note that Eq. (8) implies, by the Baker-Campbell-Hausdorff formula,

$$\exp\left[\int dy\,\Omega(x-y)J_0(y)\right]\phi(z)\exp\left[-\int dy\,\Omega(x-y)J_0(y)\right]=\exp[\Omega(x-z)]\phi(z)\,. \tag{11}$$

Using this formula the n-particle state functional may be expressed as

$$|\psi_n\rangle=\exp\left[-\sum_{j=1}^{n}\sum_{i=1}^{j-1}\Omega(x_i-x_j)\right]\left\{\exp\left[\sum_{i=1}^{n}\int dy\,\Omega(x_i-y)J_0(y)\right]\prod_{i=1}^{n}\bar\phi(x_i)|0\rangle\right\}, \tag{12}$$

where

$$\bar\phi(x)=\exp\left[i\int^{x}dy_i A_i(y)\right]\phi(x)\,.$$

Now the general form of the n-particle state functional following from the representation theory of the braid group is given by

$$\psi_S[\chi(x_1),\ldots,\chi(x_n);t]=\exp\left[-2iS\sum_{j=1}^{n}\sum_{i=1}^{j-1}\omega(x_i-x_j)\right]\psi_0[\chi(x_1),\ldots,\chi(x_n);t]\,, \tag{13}$$

where it was shown by Forte and Jolicœur [5] that for CS theory with matter coupling, the generalized spin factor S is a function of the parameter θ, henceforth denoted by $S(\theta)$. For the Klein-Gordon field $S(\theta)=1/\theta$. It may be observed, however, that the explicit functional form of $S(\theta)$ is immaterial for the ensuing analysis. $\psi_0[\chi(x_1),\ldots,\chi(x_n);t]$ is an n-particle state functional with Bose statistics and $\omega(x-y)$ is the multivalued polar angle of the vector $x-y$,

$$\omega(x-y)=\arctan\frac{x^2-y^2}{x^1-y^1}\,.$$

Going back to Eq. (12) we observe that the expression in the curly bracket represents a gauge-invariant functional with commuting one-particle cocycles, because

$$\left[\int dy\,\Omega(x-y)J_0(y),\int dy'\,\Omega(x'-y')J_0(y')\right]=0\,,$$

and hence may be identified with ψ_0 (13). The correspondence between Eqs. (12) and (13) becomes complete if one substitutes

$$\Omega(x_i-x_j)=2iS(\theta)\omega(x_i-x_j)\,.$$

Hence the final expression for the one-particle creation operator (9) is

$$\hat\phi(x)=\exp\left[2iS(\theta)\int dy\,\omega(x-y)J_0(y)+i\int^{x}dy_i A_i(y)\right]\phi(x)\,, \tag{14}$$

which is multivalued due to the presence of $\omega(x-y)$. This is our anyonic field operator since it creates states (12) with arbitrary spin $S=S(\theta)$ when acting on the vacuum. As happens for the Klein-Gordon field we may regard $\hat\phi$ as comprising creation operators of particles and annihilation operators of antiparticles. This does not affect our analysis since the n-particle functional (13) is written for distinct positions x_i only so that the vacuum contributions [proportional to $\delta(x_i-x_j)$, $i\neq j$] vanish. Similarly one-antiparticle states are created by $\hat\phi^\dagger(x)$.

The anyon operator (14) given here is different from previous constructions [1,6,7], being manifestly gauge invariant. It may be recalled that the usual anyon operators found in the literature [1,5,7], which are gauge dependent, are obtained in the Hamiltonian formalism with a specific gauge choice. It is not clear, therefore, whether their anyonicity is a physical effect or an artifact of the gauge.

To study the statistics of $\hat\phi(x)$ (14) we compute the product $\hat\phi(x)\hat\phi(y)$ and exploit the formula (11) to obtain

$$\hat\phi(x)\hat\phi(y)=e^{\pm 2i\pi S(\theta)}\hat\phi(y)\hat\phi(x)\,,$$

since $\omega(x-y)-\omega(y-x)=\pm\pi$. The sign ambiguity in the phase arises because the function ω is only defined mod(2π). Physically it reflects the arbitrariness present in the exchange of two particles which may be done via either a clockwise or an anticlockwise rotation. The above equation reveals that the fields obey graded commutation relations. For integral values of $S(\theta)$ (corresponding to bosons), commutators are obtained, while half-integral values of $S(\theta)$ (corresponding to fermions) yield anticommutators. Thus the usual spin-statistics theorem valid for bosons and fermions is reproduced.

One can similarly compute the algebra for the one-antiparticle creation operator,

$$\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)=e^{\pm 2i\pi S(\theta)}\hat{\phi}^\dagger(y)\hat{\phi}^\dagger(x)\,,$$

as well as for the particle-antiparticle case,

$$\hat{\phi}(x)\hat{\phi}^\dagger(y)=e^{\mp 2i\pi S(\theta)}\hat{\phi}^\dagger(y)\hat{\phi}(x)\,.$$

There is a subtle aspect in the definition of the anyon operator $\hat{\phi}$ (14). If the solution of the constraint (4) is substituted in (14) then $\hat{\phi}(x)$ becomes single valued and cannot represent an anyon operator. The paradox can be resolved by recalling that the constraint (4) is *first* class and, in our approach, is obeyed only weakly, in contrast to the second-class constraint P_i (2) which is strongly valid. Hence the constraint (4) cannot be directly substituted in (14).

To further understand the implications of the construction (14), we consider the interaction piece of the Hamiltonian and reexpress it in terms of the careted variables (14). We obtain

$$(D_i\phi)^*(D_i\phi)=\partial_i\left[\exp\left(-2iS(\theta)\int dy\,\omega(x-y)\hat{J}_0(y)\right)\hat{\phi}\right]^*\partial_i\left[\exp\left(-2iS(\theta)\int dy\,\omega(x-y)\hat{J}_0(y)\right)\hat{\phi}\right].$$

We observe that the explicit dependence on the potential has been eliminated by the use of the careted variables.

To conclude, we have shown that the CS theory coupled to complex scalars can be systematically quantized in the canonical formalism without gauge constraints. All the space-time symmetries are preserved and the full Poincaré algebra is valid. Our analysis naturally leads to the construction of multivalued anyonic operators which create physical states with arbitrary spin. These operators satisfy graded commutation relations which are compatible with the spin-statistics theorem. The anyonic operators found here are completely new and improve upon the previous constructions [1,6,7] since these operators are gauge independent and so the observed effects are physical. The earlier papers [1,2,6,7], however, employ specific gauge-fixing techniques to discuss anyonicity of gauge-dependent objects. Consequently their interpretation remains obscure. Moreover we have avoided the usual formal manipulations, which have led to controversies and criticisms [2-5], in obtaining the anyonic operators. The anyonic operators effectively eliminate the potential from the Hamiltonian. The extension of our analysis for fermionic matter couplings and the effects of including a Maxwell term in the theory will be discussed elsewhere.

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[1] G. Semenoff, Phys. Rev. Lett. **61**, 517 (1988); see also G. Semenoff and P. Sodano, Nucl. Phys. **B328**, 753 (1989).
[2] G. Semenoff, Phys. Rev. Lett. **63**, 1026 (1989).
[3] C. Hagen, Phys. Rev. Lett. **63**, 1025 (1989).
[4] R. Jackiw and S. Y. Pi, Phys. Rev. D **42**, 3500 (1990).
[5] S. Forte and T. Joliceur, Nucl. Phys. **B350**, 589 (1991); see also S. Forte, Rev. Mod. Phys. **64**, 193 (1992).
[6] T. Matsuyama, Phys. Lett. B **228**, 99 (1989); Prog. Theor. Phys. **84**, 1220 (1990).
[7] A. Foerster and H. O. Girotti, Phys. Lett. B **230**, 83 (1989); Nucl. Phys. **B342**, 680 (1990).
[8] P. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, NY, 1964).
[9] See, for example, K. Sundermeyer, in *Constrained Dynamics*, Lecture Notes in Physics Vol. 169, edited by H. Araki *et al.* (Springer-Verlag, Berlin, 1982).
[10] For some examples of this approach, see A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Academia Nazionale Dei Lincei, Roma, 1976).