# Energy Dissipation in Shear Driven Turbulence

Charles R. Doering [a]
*Department of Physics and Institute for Nonlinear Studies, Clarkson University, Potsdam, New York, 13699-5820*

Peter Constantin [b]
*Department of Mathematics, University of Chicago, Chicago, Illinois 60637*
(Received 21 February 1992; revised manuscript received 13 August 1992)

The Navier-Stokes equations are utilized to derive upper bounds on the turbulent energy dissipation rate for an incompressible Newtonian fluid confined between parallel comoving plates. These estimates provide a rigorous foundation for one of the basic scaling ideas of turbulence theory, namely, the independence of the dissipation rate and the viscosity at high Reynolds number. The bounds are compared to experiments on turbulence in the Couette-Taylor system.

PACS numbers: 47.25.Jn, 03.40.Gc, 47.20.Ft, 47.25.Ei

The analysis of fluid turbulence presents one of the great challenges to theoretical physics and applied mathematics. Along with the absence of analytic solutions for turbulent flows there are still unresolved fundamental mathematical issues: It is an open question whether or not solutions of the 3D Navier-Stokes equations lose regularity at high Reynolds numbers [1]. Recent years have witnessed significant advances in the analysis of the 2D Navier-Stokes equations, notably the rigorous and accurate estimation of the scaling of strange attractor dimensions (predominantly) for body-force driven flows [2]. Flows driven by boundary conditions, like shear flows or convection, are frequently encountered in applications but present some technical challenges [3]. Of considerable recent interest both theoretically and experimentally is the phenomenon of scaling [4]—or the lack of scaling [5]—in the global properties of such flows.

In this Letter we present rigorous upper bounds on the time-averaged energy dissipation rate for an incompressible Newtonian fluid subject to a boundary-induced shear. We show that the hypothesis of high Reynolds number independence of the turbulent energy dissipation rate and the viscosity [6] holds as an upper bound for this geometry. Our analysis, introducing the "background flow" method, makes no *a priori* assumptions about the flow or its spectrum and we use only elementary functional estimates. These estimates hold for the high Reynolds numbers weak solutions of the 3D Navier-Stokes equations (see Ref. [1] for details on the distinction between weak and strong solutions) and they are of fundamental importance because they provide a rigorous connection between the scaling hypothesis and the mathematical model for incompressible Newtonian fluids provided by the Navier-Stokes equations.

The calculation is presented in detail below both because of its simplicity and because the technique is applicable to other boundary driven flows. In the conclusions we interpret our results in terms of experiments on turbulent flow between concentric cylinders (the Couette-Taylor geometry) and point out some directions for further development of the background flow method.

We base our analysis on the incompressible Navier-Stokes equations,

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\nu\Delta\mathbf{u}\,, \tag{1a}$$

$$\nabla\cdot\mathbf{u}=0\,, \tag{1b}$$

for the velocity vector field $\mathbf{u}=(u_x,u_y,u_z)=(u_1,u_2,u_3)$, where $\mathbf{x}=(x,y,z)=(x_1,x_2,x_3)$, $\nu$ is the kinematic viscosity, and $p$ is the pressure determined by the incompressibility condition. Mass units are chosen so that the density $\rho=1$. The fluid is contained between rigid parallel plates located at $z=0$ and $z=h$. The $x$ coordinate lies between 0 and $L_x$, the $y$ coordinate lies between 0 and $L_y$, and we impose periodic boundary conditions in the $x$ and $y$ directions. The fluid is driven by the boundary at $z=0$ moving in the $x$ direction at speed $U$: $\mathbf{u}(x,y,0,t)=\mathbf{i}U$ and $\mathbf{u}(x,y,h,t)=0$. The setup is illustrated in Fig. 1. The Reynolds number is defined as $R=Uh/\nu$.

The average energy dissipation rate (per unit volume) is

$$\varepsilon=\nu\langle\|\nabla\mathbf{u}\|_2^2\rangle/L_xL_yh\,, \tag{2}$$

where [7]

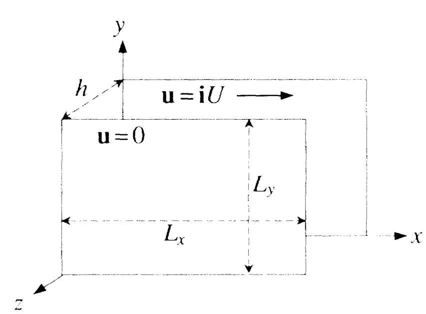


FIG. 1. Plates of dimension $L_x\times L_y$ are separated by gap $h$ in the $z$ direction. The plate at $z=0$ is moving at speed $U$ in the $x$ direction and the plate at $z=h$ is stationary. Boundary conditions are periodic in the $x$ and $y$ directions.

$$||\nabla \mathbf{u}||_2^2 = \sum_{i,j=1}^{3} \left|\left| \frac{\partial u_i}{\partial x_j} \right|\right|_2^2 , \tag{3}$$

and $\langle \cdots \rangle$ denotes the time average [8]

$$\langle f(t) \rangle = \overline{\lim_{t\to\infty}} \frac{1}{T} \int_0^T dt\, f(t) . \tag{4}$$

The goal is to establish rigorous bounds on $\varepsilon$ in terms of the system parameters $(U,\nu,h,L_x,L_y)$ directly from the Navier-Stokes equations. For example, the dissipation rate for the stationary laminar solution, $\mathbf{u}_{\text{laminar}} = \mathbf{i}U(1-z/h)$, is a lower bound on $\varepsilon$:

$$\varepsilon \geq \varepsilon_{\text{laminar}} = \nu U^2/h^2 . \tag{5}$$

In the nonstationary case the total kinetic energy must be controlled in order to take the time average. Taking the scalar product $\mathbf{u}$ with the Navier-Stokes equation and integrating over all space we find, via appropriate integrations by parts, the energy equation

$$\frac{d}{dt}\frac{1}{2}||\mathbf{u}||_2^2 = -\nu||\nabla \mathbf{u}||_2^2 - \nu U \int_0^{L_x} dx \int_0^{L_y} dy \left. \frac{\partial u_x}{\partial z} \right|_{z=0} . \tag{6}$$

Assuming that $||\mathbf{u}(\cdot,t)||_2^2$ is bounded uniformly in time, the time average of this equation yields

$$\nu\langle ||\nabla \mathbf{u}||_2^2 \rangle = -\nu U \int_0^{L_x} dx \int_0^{L_y} dy \left. \left\langle \frac{\partial u_x}{\partial z} \right\rangle \right|_{z=0} . \tag{7}$$

This identity says only that the rate of energy dissipation by the fluid viscosity is the power expended by the agent enforcing the boundary conditions while working against the drag at the boundary. It does not allow us to estimate $\varepsilon$ solely in terms of the system parameters. That is, it is not a closed relationship for $\varepsilon$ because we cannot *a priori* connect the $L^2$ norm of the gradient of $\mathbf{u}$ to the shear rate on the boundary. Moreover, it has not been established that the total energy is uniformly bounded in time.

The problems with the boundary term may be circumvented by changing variables. Decompose the velocity (in the spirit of the conventional decomposition into mean and fluctuating components) as

$$\mathbf{u}(x,y,z,t) = \mathbf{i}\phi(x) + \mathbf{v}(x,y,z,t) , \tag{8}$$

where $\mathbf{u}$'s boundary conditions are contained in the "background flow" $\mathbf{i}\phi(x)$: $\phi(0)=U$ and $\phi(h)=0$. Then $\mathbf{v}$ satisfies

$$\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{i}\phi' v_z + \phi\frac{\partial}{\partial x}\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} + \mathbf{i}\nu\phi'' , \tag{9a}$$

$$\nabla\cdot\mathbf{v} = 0 , \tag{9b}$$

where $\phi'$ and $\phi''$ denote derivatives of $\phi$. The boundary conditions for $\mathbf{v}$ are periodic in the $x$ and $y$ directions while in the $z$ direction, $\mathbf{v}(x,y,0,t) = 0 = \mathbf{v}(x,y,h,t)$. The energy equation for $\mathbf{v}$, obtained by dotting $\mathbf{v}$ into Eq. (9a), integrating, and integrating by parts, is

$$\frac{d}{dt}\frac{1}{2}||\mathbf{v}||_2^2 = -\nu||\nabla\mathbf{v}||_2^2 - \int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, \phi' v_x v_z - \nu\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, \phi' \frac{\partial v_x}{\partial z} . \tag{10}$$

The driving comes from the last two terms on the right, one quadratic and one linear in $\mathbf{v}$. The background flow profile is essentially arbitrary at this stage, constrained only by the boundary conditions.

Consider first the choice $\mathbf{i}\phi(z) = \mathbf{u}_{\text{laminar}}$. Then the linear driving term is absent and for low enough Reynolds numbers we may explicitly bound the energy in $\mathbf{v}$. According to the Schwarz inequality [9] and the relation $2ab \leq a^2 + b^2$,

$$\left| \int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, v_x v_z \right| \leq ||v_x||_2 ||v_z||_2 \leq \tfrac{1}{2}||\mathbf{v}||_2^2 . \tag{11}$$

Because each component of $\mathbf{v}$ is periodic in $x$ and $y$ and vanishes at $z=0$ and $h$, Poincaré's inequality [10] implies

$$||\nabla\mathbf{v}||_2^2 \geq (\pi^2/h^2)||\mathbf{v}||_2^2 . \tag{12}$$

Then

$$\frac{d}{dt}\frac{1}{2}||\mathbf{v}||_2^2 \leq -\frac{\nu}{2h^2}(2\pi^2 - R)||\mathbf{v}||_2^2 . \tag{13}$$

Thus if $R < 2\pi^2 \sim 20$, then the energy in the perturbation decays exponentially with time [11] and the laminar solution is nonlinearly stable. When $R > 2\pi^2$ we may conclude only that the kinetic energy grows at most exponentially in time.

The technical problem for large $R$ is that both the dissipative and driving terms are quadratic in $\mathbf{v}$ so a restriction on the *coefficients* of these terms is necessary for the dissipation to control the energy input. To proceed we are compelled to choose a background flow which satisfies the boundary conditions and which drives $||\mathbf{v}||_2^2$ with a "small" quadratic term, although perhaps with a "large" term of lower order in $\mathbf{v}$ [12]. This is accomplished with the functional form

$$\phi(z) = \begin{cases} (U/2\delta)(2\delta - z), & 0 \leq z \leq \delta , \\ U/2, & \delta \leq z \leq h-\delta , \\ (U/2\delta)(h-z), & h-\delta \leq z \leq h , \end{cases} \tag{14}$$

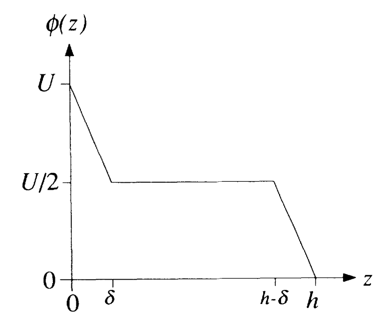


FIG. 2. Background flow profile $\phi(z)$. This profile is *not* a solution of the Navier-Stokes equation, nor does it necessarily correspond to a mean flow profile.

as illustrated in Fig. 2. We refer to the parameter $\delta$ as the "boundary layer thickness." The advantage of this form is that it concentrates the support of $\phi'$ near the boundaries where the components of $\mathbf{v}$ vanish.

Application of the fundamental theorem of calculus and the Schwarz inequality shows that the $x$-$y$ integral of the product $v_x v_z$ is bounded uniformly in $z$ according to

$$\left|\int_0^{L_x} dx \int_0^{L_y} dy\, v_x(x,y,z) v_z(x,y,z)\right| \le z \left[\int_0^{L_x} dx \int_0^{L_y} dy \int_0^z dz \left(\frac{\partial v_x}{\partial z}\right)^2\right]^{1/2} \left[\int_0^{L_x} dx \int_0^{L_y} dy \int_0^z dz \left(\frac{\partial v_z}{\partial z}\right)^2\right]^{1/2}. \quad (15)$$

An analogous estimate holds near the $z=h$ boundary. The quadratic source term [the next-to-last term in Eq. (10)] is then simply estimated in terms of $\delta$ and the dissipation:

$$\left|\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, \phi' v_x v_z\right| = \frac{U}{2\delta}\left|\int_0^{L_x} dx \int_0^{L_y} dy \int_0^\delta dz\, v_x v_z + \int_0^{L_x} dx \int_0^{L_y} dy \int_{h-\delta}^h dz\, v_x v_z\right|$$

$$\le \frac{U\delta}{4}\left\{\frac{1}{2\sqrt{2}}\left\|\frac{\partial v_x}{\partial z}\right\|_2^2 + \frac{\sqrt{2}}{2}\left\|\frac{\partial v_z}{\partial z}\right\|_2^2\right\} \le \frac{U\delta}{8\sqrt{2}}\|\nabla \mathbf{v}\|_2^2, \quad (16)$$

where the incompressibility constraint on $\mathbf{v}$ was used in the last step above [13]. The linear source term [the last term in Eq. (10)] is also simply estimated:

$$\left|\nu\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, \phi' \frac{\partial v_x}{\partial z}\right| \le \nu\|\phi'\|_2 \left\|\frac{\partial v_x}{\partial z}\right\|_2 \le \nu U\left(\frac{L_x L_y}{2\delta}\right)^{1/2} \|\nabla \mathbf{v}\|_2. \quad (17)$$

Injecting the bounds in Eqs. (16) and (17) into the energy equation (10),

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{v}\|_2^2 \le -\nu\|\nabla\mathbf{v}\|_2^2 + \frac{U\delta}{8\sqrt{2}}\|\nabla\mathbf{v}\|_2^2 + \nu U\left(\frac{L_x L_y}{2\delta}\right)^{1/2}\|\nabla\mathbf{v}\|_2. \quad (18)$$

The boundary layer thickness in the background flow may now be adjusted so that the viscous dissipation overcomes the quadratic driving term. We choose

$$\delta = 4\sqrt{2}(\nu/U) = 4\sqrt{2}hR^{-1}. \quad (19)$$

Using Poincaré's inequality and $2ab \le a^2+b^2$ to break up the last term, we find

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{v}\|_2^2 \le -\frac{\nu\pi^2}{4h^2}\|\mathbf{v}\|_2^2 + \frac{\nu U^2 L_x L_y}{2\delta}. \quad (20)$$

Hence the kinetic energy decreases if it is greater than $U^2h^2L_xL_y/\pi^2\delta$ and we conclude that the kinetic energy is indeed bounded uniformly in time.

The time-averaged energy dissipation rate is bounded by taking the time average of Eq. (10) and using Eq. (16) combined with the definition of $\delta$ in Eq. (19) to see that

$$\frac{\nu}{2}\langle\|\nabla\mathbf{v}\|_2^2\rangle + \nu\left\langle\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, \phi' \frac{\partial v_x}{\partial z}\right\rangle = -\frac{\nu}{2}\langle\|\nabla\mathbf{v}\|_2^2\rangle - \left\langle\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, \phi' v_x v_z\right\rangle \le 0. \quad (21)$$

Then

$$\varepsilon = \frac{\nu}{L_xL_yh}\left\{\langle\|\nabla\mathbf{v}\|_2^2\rangle + 2\left\langle\int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz\, \phi' \frac{\partial v_x}{\partial z}\right\rangle + \|\phi'\|_2^2\right\}$$

$$\le \frac{\nu}{L_xL_yh}\|\phi'\|_2^2$$

$$= \frac{1}{8\sqrt{2}}\frac{U^3}{h}. \quad (22)$$

This rigorous upper bound on the energy dissipation (valid when $R \ge 8\sqrt{2}$ so that $\delta \le h/2$) is *independent of the viscosity*, in accord with the scaling view of turbulent energy dissipation.

We may interpret this result for the case of turbulent flow between concentric cylinders, at least in the limit of large aspect ratio and a narrow gap. The average torque $G$ (measured in units of $\rho\nu^2 L_y$, where $L_y$ corresponds to the length of the cylinders) required to rotate the inner cylinder at angular speed $2\pi U/L_x$ (where $L_x$ corresponds to the circumference of the inner cylinder) is the total-energy dissipation rate ($\varepsilon \times L_xL_yh$) divided by the rotation rate. Utilizing the bound in Eq. (22),

$$G = \frac{\varepsilon L_xL_yh}{(2\pi U/L_x)\rho\nu^2L_y} \le 0.0141\left(\frac{L_x}{h}\right)^2 R^2. \quad (23)$$

In Ref. [5], Lathrop, Fineberg, and Swinney report

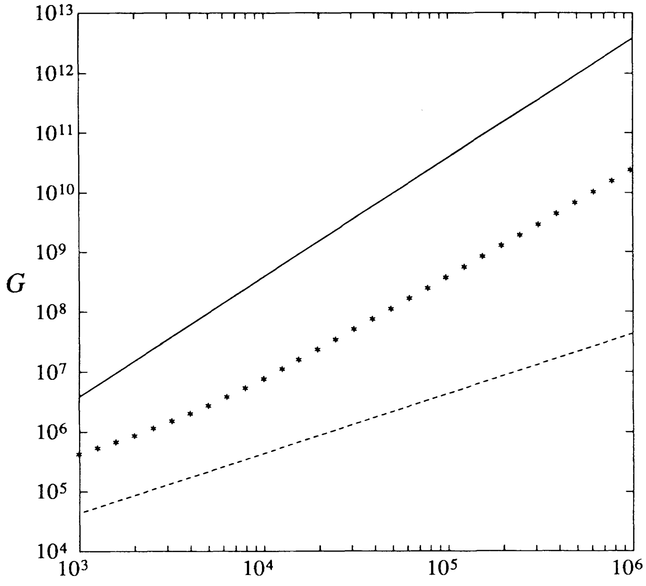


FIG. 3. Torque vs Reynolds number for turbulent flow between concentric cylinders. Solid line: upper bound from Eq. (23). Dashed line: lower bound using Eq. (5). Discrete points: fit to experimental data of Ref. [5]. The ratio $L_x/h=16.5$, from Ref. [5], is used in the bounds.

measurements of $G$ vs $R$ for Reynolds numbers up to $10^6$. In Fig. 3 we plot their torque data along with the upper bound in Eq. (23) and the lower bound $[G \geq (2\pi)^{-1} \times (L_x/h)^2 R]$ computed from Eq. (5). We observe that the rigorous upper bound is 1 to 2 orders of magnitude above the experimental values. Experimentally it is found that the exponent $\alpha$ in the scaling relation $G \sim R^{\alpha}$ is not constant, but increases from 1.66 to 1.87 as $R$ increases between $\sim 1.3\times 10^4$ and $10^6$. Extrapolated, the data suggest that $\alpha$ will reach the value 2 at $R \approx 1.5\times 10^7$. The bound derived here implies that $\alpha$ cannot continue to increase as $R$ increases, but that it must eventually stay at or below the scaling limit $\alpha=2$.

We close with several comments. It is an open question whether our upper bound for the energy dissipation rate can be improved by utilization of a more sophisticated background flow ansatz, or whether scaling ever sets in for such a wall bounded shear flow (the phenomenological *logarithmic friction law* [14] predicts otherwise). The background flow method can also be applied to other systems such as pipe flow and thermal convection in the Boussinesq approximation [15], and the bounds derived for those cases, in addition to that derived in this paper, are in accord with the scaling obtained by a variational approach [16]. It remains a challenge to apply the background flow method to open systems like grid-generated turbulence—where experiments anticipate scaling of the energy dissipation rate like that found here [17]—or for flow past a solid object.

We thank J. D. Gibbon, L. P. Kadanoff, D. P. Lathrop, C. D. Levermore, B. Luce, M. Orwoll, T. S. Ray, K. R. Sreenivasan, and H. L. Swinney for helpful discussions. C.R.D. is supported by NSF Grants No. PHY-8958506 and No. PHY-8907755, and acknowledges the hospitality of the Antelope Retreat Center where part of this work was completed. P.C. is supported by NSF Grant No. DMS-8903244.

(a)Electronic address: doering@craft.camp.clarkson.edu.
(b)Electronic address: const@zaphod.uchicago.edu.

[1] P. Constantin and C. Foias, *The Navier-Stokes Equations* (University of Chicago Press, Chicago, 1988).
[2] P. Constantin, C. Foias, and R. Temam, Physica (Amsterdam) **30D**, 284 (1988); C. R. Doering and J. D. Gibbon, Physica (Amsterdam) **48D**, 471 (1991).
[3] See, for instance, R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics* (Springer-Verlag, Berlin, 1988).
[4] F. Heslot, B. Castaing, and A. Libchaber, Phys. Rev. A **36**, 5870 (1987).
[5] D. P. Lathrop, J. Fineberg, and H. L. Swinney, Phys. Rev. Lett. **68**, 1515 (1992).
[6] G. I. Taylor, Proc. R. Soc. London A **151**, 421 (1935); A. N. Kolmogorov, C. R. (Dokl.) Acad. Sci. URSS **30**, 301 (1941).
[7] The $L^2$ norm (squared) of a function $f(x,y,z)$ is defined by $||f||_2^2=\int |f(x,y,z)|^2 dx\,dy\,dz$.
[8] The symbol $\overline{\lim}$ means limit supremum. We take the largest possible value for the time average.
[9] The Schwarz inequality states that $|\int fg| \leq (\int |f|^2)^{1/2} \times (\int |g|^2)^{1/2}$.
[10] Poincaré's inequality asserts that the $L^2$ norm squared of the gradient of a function satisfying some boundary conditions is greater than the norm squared of the function times the smallest eigenvalue of $-\Delta$ with those boundary conditions. Here, that smallest eigenvalue is $\pi^2/h^2$.
[11] Gronwall's inequality implies that if $f \geq 0$ and $df/dt \leq -\gamma f$, then $f(t) \leq f(0)\exp\{-\gamma t\}$.
[12] E. Hopf, in *Lecture Series of the Symposium on Partial Differential Equations, Berkeley, 1955* (University of Kansas Press, Lawrence, 1957).
[13] $\nabla\cdot \mathbf{v}=0$ implies that $||\partial v_z/\partial z||_2^2 \leq \frac{1}{2}\{||\partial v_z/\partial z||_2^2+||\partial v_x/\partial x||_2^2+||\partial v_y/\partial y||_2^2+||\partial v_x/\partial y||_2^2+||\partial v_y/\partial x||_2^2\}$.
[14] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT, Cambridge, MA, 1972).
[15] P. Constantin and C. R. Doering (unpublished).
[16] L. N. Howard, J. Fluid Mech. **17**, 405 (1963); F. H. Busse, J. Fluid Mech. **37**, 457 (1969); E. C. Nickerson, J. Fluid Mech. **38**, 807 (1969).
[17] K. R. Sreenivasan, Phys. Fluids **27**, 1048 (1984).