Transition to Dissipation in a Model of Superflow

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A direct numerical study of a model of superflow, the nonlinear Schrödinger equation, and simple analytical arguments shows a very striking phenomenon: The flow around a disk creates a drag force beyond a well-defined threshold velocity, linked to the emission of vortices from the perimeter of the disk.

It has been proposed [1] to describe neutral superflows by the nonlinear Schrödinger equation (NLSE), which reads in a dimensionless form as

$$i\frac{\partial \Psi}{\partial t} = \frac{1}{2}\Delta\Psi - |\Psi|^2\Psi. \tag{1}$$

The Madelung transformation maps the NLSE into a fluidlike equation, $\rho = |\Psi|^2$ being the density, although the fluid velocity is minus the gradient of the phase ϕ of Ψ:

$$\frac{\partial \rho}{\partial t} = \nabla\cdot(\rho\nabla\phi), \tag{2a}$$

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2\rho^{1/2}}\nabla^2(\rho^{1/2}) + \frac{1}{2}(\nabla\phi)^2 + \rho. \tag{2b}$$

It has long been known too that the phase singularities of Ψ, at the zeros of this function, play a role very similar to the one of point vortices in ordinary inviscid hydrodynamics (perfect fluid), an analogy that has been well studied [2]. Note that these phase singularities are not singular in a mathematical sense as far as the NLSE is concerned. However, there are differences between perfect fluids and what is described by the NLSE. Indeed one of these differences is the quantization of the circulation in vortices of the NLSE, with no counterpart in ordinary fluids. Another difference, perhaps less obvious, is that the NLSE bears sound waves of finite speed, although one usually makes a comparison between the NLSE and incompressible fluids. We show here that the mixing of the two phenomena, compressibility and quantization of circulation, has important consequences that show up very dramatically in computer simulations. Our findings are based upon the comparison between simple theoretical arguments and numerical simulations of the NLSE. Jones and Roberts [3] have studied numerically solutions of the NLSE in the form of ring vortices assumed to be stationary in a moving frame of reference, and in an infinite system without any obstacle. Our simulations yield a time evolution of solutions of the NLSE aimed at modeling a flow around a fixed obstacle. Vortices, when they appear in our simulations, are produced by the system itself, without being put in at the beginning. Our central result concerns the drag on a moving object. It happens that this drag is zero until a critical velocity, where it begins to increase. In an ordinary perfect fluid, this critical velocity would be the one where a shock begins to form, but here no shock wave can exist because of the absence of intrinsic molecular damping, so that shocks are replaced by the emission of vortices, which leads to a vortical and dissipative flow on large scales.

Although formally simple, the NLSE has a number of nontrivial features. A crucial one is the Galilean invariance. Given in free space a solution $\Psi(\mathbf{x},t)$, with $\mathbf{x}$ space and t time, then $\Psi(\mathbf{x}-\mathbf{v}t,t)\exp[i(\mathbf{v}\cdot\mathbf{x}-\frac{1}{2}v^2t)]$ is still a solution, where $\mathbf{v}$ is the velocity of an arbitrary Galilean boost. Another important property of the NLSE is its phase dynamics: The NLSE is invariant under a global phase change $\Psi\rightarrow\Psi e^{i\phi}$, ϕ arbitrary real. This implies that long-wavelength perturbations of the phase have to decay slowly. In particular, when linearized around a uniform solution, $\Psi_0 = \rho^{1/2}e^{i\rho t}$, the phase equation takes [4] the form $\partial^2\phi/\partial t^2 = c_s^2\Delta\phi$, with $c_s = \rho^{1/2}$. This has two consequences: First, the time-independent flows are irrotational, because then the phase equation becomes simply Laplace's equation $\Delta\phi = 0$, the fluid velocity being $-\nabla\phi$. Then, for time-dependent flows there is a finite speed of propagation of perturbations, $\rho^{1/2}$. A very important question in superfluidity is the limit speed: Below a critical speed there is no dissipation, and one appears beyond this critical speed. To check whether this transition is present in the NLSE, we did the following computer experiment: We started with initial conditions with a fixed disk in a large box [5], and then made a Galilean boost of the solution by keeping the disk fixed. The boundary condition on the disk perimeter is the Dirichlet condition $\Psi = 0$, a crude modelization indeed of the interaction between a solid surface and the condensate. It represents very roughly the fact that the superfluid does not flow into the solid. More complicated boundary conditions have been proposed [6]. Although we have not made numerical simulations with these elaborate conditions, it is likely that the phenomena discussed below are not dependent on the details of the boundary conditions. In particular, the loss of steady solution beyond a certain value of the flow speed is unaffected by the details of the boundary condition on the solid, as long as the velocity is tangent to this boundary. The details of this boundary condition

should be relevant at length scales of the order of the size of the vortex core, and so could affect, for instance, the details of the release of the vortices by the solid boundary, but not the frequency, or the drag law. The relevant parameter for molecular effects is the ratio of the vortex core to the diameter of the disk, assumed to be small here ($\frac{1}{10}$ in our numerical simulations).

At small velocities, the solution accommodates the boundary conditions everywhere after some transient to yield a stationary velocity field without dissipation, because of the d'Alembert paradox for perfect fluids. Beyond a critical velocity, that we shall characterize later, vortices begin to be emitted [7] more or less periodically from the disk, yielding an average drag on the disk, this drag tending linearly to zero at the onset, as does the frequency of emission of vortices. We can estimate the drag from the energy transferred by the flow to the disk. This energy for one period of emission of vortices is $E_{\text{flow}} = F\langle v \rangle T$, where F is the mean drag, $\langle v \rangle$ the mean flow velocity, and T the period. The energy of the vortex pair is $E_{\text{vortex}} = \Gamma^2 \ln(2a)$, where a is the diameter of the disk, in the microscopic unit length used to write the NLSE in the dimensionless form (1), and Γ is the quantized circulation, equal to $+1$ or -1 with our units. The period T diverges as the velocity at infinity tends to the critical velocity from above. Let $\varepsilon = v - v_c$ be the small difference between the actual value of the velocity at infinity and its value at the threshold of emission of vortices (or of drag). The quantity ε is chosen for measuring the velocity at infinity in such a way that the transition to drag occurs at $\varepsilon = 0$: There is no permanent drag for ε negative and there is such a drag for ε positive. After the emission of a vortex, its velocity field balances the main velocity field to make it locally less than the critical value everywhere on the disk. As this vortex is convected downstream, however, its contribution to the velocity on the perimeter of the disk decreases until the total velocity there again becomes larger than the critical value, triggering the emission of a new vortex, and so on. As the velocity induced by a vortex decreases like the inverse of the distance, a simple estimate shows that $\varepsilon = \Gamma/\langle v \rangle T$; then $T \sim 1/\varepsilon$ and $F \sim \varepsilon$. At higher velocities, the emission of vortices becomes more and more irregular with an increasing average frequency and the vortices form a kind of turbulent wake. In the far wake these vortices radiate their potential energy into the phonon field, due to the acceleration they are submitted to in their rather chaotic motion (interaction of vortices of like sign, with the cylinder, and so on), and all energy and momentum are ultimately carried away at long distances by phonons. Lund has calculated the loss of interaction energy of accelerated vortices [8] due to this radiation in a compressible fluid.

The transition to time-dependent flow can be understood as follows: In a perfect incompressible flow around a disk, the fluid velocity is the greatest at the point across

the stream velocity on the perimeter, where the fluid velocity is twice that at infinity. Here the flow becomes supersonic when the fluid velocity is only half the sound speed at infinity. In ordinary fluids this would lead to a shock wave. But here this is impossible, as shock waves require dissipation in a thin layer, and there is no such formally dissipative term in the NLSE. Instead, in the supersonic region, vortices are generated on the perimeter, very near the point of maximum velocity, as seen in Fig. 1. On a large scale a string of vortices with the same sign may be seen as a tangential discontinuity (or shear layer) of the fluid velocity. This tangential discontinuity allows the matching of different domains of fluid with different velocities, and so avoids the formation of shocks, which normally are the discontinuities instead. All of this can be made more rigorous by trying to find a stationary solution of the NLSE at distances from the surface much greater than the microscopic unit length, and with a uniform flow at infinity. Putting $\partial \rho / \partial t = 0$ in (2a) we obtain the continuity equation $\nabla \cdot (\rho \nabla \phi) = 0$, and ρ is determined by (2b) together with $\partial \phi / \partial t = \rho_0 + v_\infty^2 / 2$, expressing the frequency of the solution of (1) at infinity. Note that a steady flow corresponds to a solution of (1) with a single uniform frequency. Neglecting $-(1/2\rho^{1/2})\nabla^2(\rho^{1/2})$ on the right-hand side of (2b) because it involves higher space derivatives that are precisely neglected in this long-wavelength approximation, one obtains from (2b)

$$\rho = \rho_0 + \frac{v_\infty^2}{2} - \frac{(\nabla \phi)^2}{2} = \rho_0 + \frac{v_\infty^2}{2} - \frac{v^2}{2}.$$
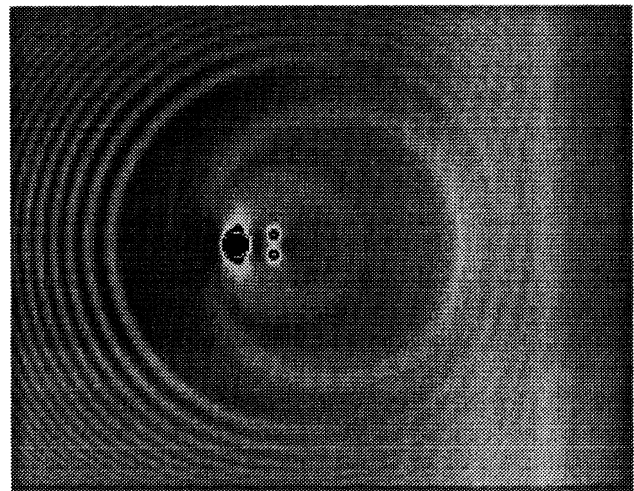

FIG. 1. Cylinder immersed in the "fluid." One represents the modulus of Ψ at an instant of time. The cylinder is the black circle in the middle, and the boundary condition on its surface is $\Psi = 0$. The speed at infinity is half the speed of sound, slightly above the onset of drag. This shows that the drag is due to the emission of vortices on the surface of the cylinder at the point of maximal fluid velocity. The vortices appear as white dots close to the cylinder and are convected by the mean flow. The sound waves seen far from the cylinder are transients, not relevant for the onset of continuous drag.

Here ρ_0 is the normal density related to the sound speed $(c_s^2 = \rho_0)$, v_∞ is the velocity at infinity, and v is the local velocity. Next if $v^2 > \rho(v)$ [i.e., $\frac{1}{2}(3v^2 - v_\infty^2) > c_s^2$] a local instability develops, leading to the release of vortices (we point out that the velocity near the vortex center is of the order of the sound speed). This implies that the steady solutions, at least for large objects, do not exist anymore when the flow becomes supersonic somewhere, and there is a critical velocity beyond which there is dissipation. Via the hodograph transformation (see Sec. 108 in Ref. [9]) one can show that the maximum velocity of a permanent flow of perfect compressibility is always reached on the perimeter of the obstacle. This maximum velocity can be approximated by $2v_\infty$, as in the incompressible case, yielding a critical velocity equal to $\frac{2}{11}^{1/2} c_s$, in agreement with the numerically observed value.

This dissipation can be measured in computer experiments by the mean drag on the disk. This drag $\mathbf{F}$ is given in general by the integral of the stress tensor on the perimeter of the disk. In Cartesian components $F_\mu = \int T_{\mu\nu} n^\nu ds$, where n^ν is the normal to the perimeter and ds the line element there. From (1), one has the conservation relations $\partial\rho/\partial t + \partial_\mu J_\mu = 0$, with $J_\mu = \frac{1}{2} i(\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*)$ and $\partial_\mu = \partial/\partial x_\mu$. The momentum conservation yields $\partial J_\mu/\partial t + \partial_\nu T_{\mu\nu} = 0$, with

$$T_{\mu\nu} = -|\Psi|^2 \delta_{\mu\nu} + \frac{1}{2}(\Psi^* \partial_{\mu\nu}\Psi + \Psi \partial_{\mu\nu}\Psi^*)$$
$$- \frac{1}{2}(\partial_\nu\Psi^* \partial_\mu\Psi + \partial_\nu\Psi \partial_\mu\Psi^*) .$$

There is no drag for a "steady" (actually with a single quantum frequency for Ψ) solution of the NLSE, another version of d'Alembert's paradox. For an unsteady solution, the time-averaged streamwise component of the drag has no reason to vanish. We plot in Fig. 2 its dependence on the flow velocity at infinity, showing its critical behavior at the onset of the release of vortices. At very large velocities, one would expect a Newton drag law (drag proportional to the square velocity), because the vortices would be then emitted more or less as a continuous line, yielding a separated wake with the same velocity as the disk in a background fluid at a different velocity, which is the familiar Kirchoff picture for the Newton drag. It is possible, however, that this is changed by turbulence in some way. We were not able, however, to compute the drag force for high velocities due to numerical instabilities for average velocities bigger than $2c_s$.

It is interesting to notice too that this might provide an explanation for the difficulty met in reaching the critical speed in superfluids: Recall first that the maximum speed is on the perimeter; then suppose that the disk perimeter is corrugated, with a length scale much smaller than the radius of the disk, but still much larger than the diameter of the vortex core (say 1-μm corrugation compared to 5-Å vortex core). Then the far field for these corrugations will already be the near field of the inviscid flow on the
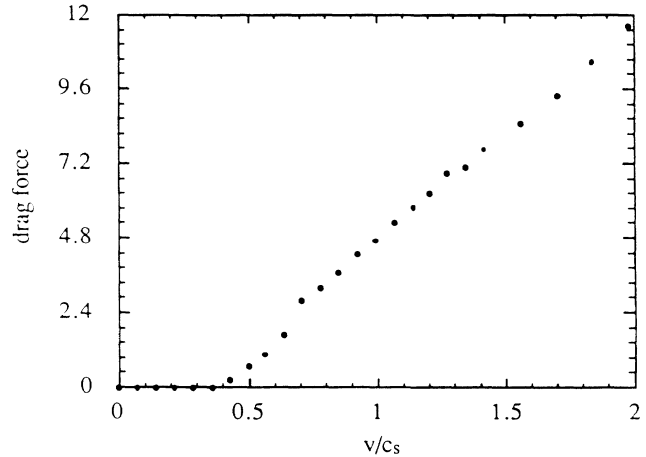

FIG. 2. Drag force on the cylinder vs the velocity of the fluid (in units of c_s).

disk. We have seen that this perimeter flow is accelerated somewhere by a factor of 2 (for a sphere this would be a factor $\frac{3}{2}$). Whence, the corrugations will likely still increase this velocity by some factor larger than 1, making the transition speed even less, when measured by the speed at infinity. In particular, this increase of the local speed is even larger near singular points of the solid surface (which could be on small crystals sitting on the surface of the vessel containing the fluid), as it is known that the fluid velocity diverges near these singularities for a perfect incompressible fluid [9].

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[5] We used a Gauss-Seidel Crank-Nicholson finite-difference method to integrate Eq. (1). The time stepping for $\Psi(t)$ from $t = 0$ to δt is as follows: Write the NLSE as $i \partial\Psi/\partial t = H(t)\Psi$, where $H(\cdot)$ is a nonlinear Hermitian operator with an obvious definition from (1). Then $\Psi(\delta t)$ is defined from $\Psi(0)$ through $[1 + \frac{1}{2} i\delta t H(\delta t)]\Psi(\delta t) = [1 - \frac{1}{2} i\delta t H(0)]\Psi(0)$. This (classical) numerical scheme yields, for δt small, $\Psi(\delta t)$ exact to second order

in δt at least, and has the very big advantage of conserving exactly the norm and energy. We took the disk diameter $= 10$, box size $= 512 \times 256$, with boundary conditions $\Psi = 0$ on the disk and no reflection on the outer boundary. We implemented the method of no reflection by artificially imposing a very large dissipation on the outer boundary. This dissipation was imposed by changing the equations gradually from conservative dynamics to dissipative dynamics when reaching farther from the "working area." In this working area (about 20 times larger than the radius of the cylinder), the equation is exactly the NLSE. The area with damping does not appear in the pictures presented. Other outer boundary conditions were implemented, like adaptative mesh in space, to increase the space step far from the working area, without changing significantly the numerical results. There is no problem for matching the outer boundary condition with the phase of the vortices at infinity, because the vortices are emitted in pairs of opposite signs and so do not alter drastically the boundary condition for the phase at infinity. The norm and the energy of the solution are conserved in time to better than 1 part per 10^8 per time step and to 1 part per 10^7 per unit time.

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[7] G. G. Nancolas *et al.*, Nature (London) **316**, 797 (1985), have observed an onset of drag on ions in superfluid helium, which they interpret, as does K. W. Schwarz [Nature (London) **316**, 766 (1985)], by the emission of ring vortices. This is very likely close to our findings. However, we show here that the mechanism of the release of vortices in the present case is quite different from the usual formation of vortices in Bénard–von Karman wakes, very crucially related to the dynamics of viscous boundary layers, totally absent in solutions of the NLSE.

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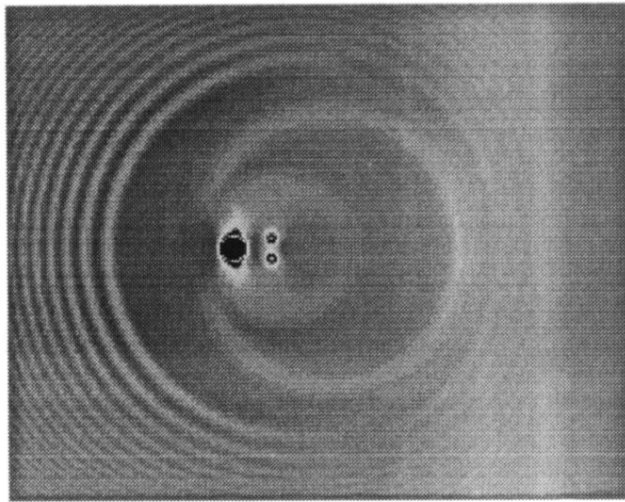FIG. 1. Cylinder immersed in the "fluid." One represents the modulus of Ψ at an instant of time. The cylinder is the black circle in the middle, and the boundary condition on its surface is $\Psi = 0$. The speed at infinity is half the speed of sound, slightly above the onset of drag. This shows that the drag is due to the emission of vortices on the surface of the cylinder at the point of maximal fluid velocity. The vortices appear as white dots close to the cylinder and are convected by the mean flow. The sound waves seen far from the cylinder are transients, not relevant for the onset of continuous drag.