Taking the Square Root of the Discrete $1/r^2$ Model

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We exhibit a factorization of the exactly solvable discrete $1/r^2$ exchange Heisenberg model. We express the Hamiltonian of the model on a lattice of L sites, as a sum over the squares of L operators in eight distinct ways, using the eight generators of the SU(3) group, and demonstrate that each of the 8 L operators annihilate the Gutzwiller wave function. The wave function is thus proven to be the exact ground state of the $1/r^2$ model, and we also provide a scheme for the construction of an infinite number of Hamiltonians for which it is the ground state.

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The one-dimensional Heisenberg antiferromagnetic model with $1/r^2$ exchange was solved by Shastry [1] and independently by Haldane [2]. The Gutzwiller wave function [3] $\Psi_g = \exp(i\pi\sum x_j)\prod_{i<j}\sin^2\phi(x_i-x_j)$, with $\phi=\pi/L$, $1\le x_i \le L$, was shown to be an exact eigenfunction of the Hamiltonian

$$H_0=\phi^2\sum_{i<j}\frac{1}{\sin^2\phi(x_i-x_j)}S_i^a S_j^a\,, \tag{1}$$

with an eigenvalue $E_0=-\phi^2 L(L^2+5)/24$. However, there has been no analytic demonstration of Ψ_g being the absolute ground state of the model. Moreover, it has not been clear whether H_0 is the only Hamiltonian for which Ψ_g is the ground state. In this work, I prove rigorously that Ψ_g is the absolute ground state of an infinite number of operators that I display explicitly, and among them is H_0. Central to my work is the surprising result that the Hamiltonian can be rewritten as a sum over squares of operators in (at least) eight different ways, with the SU(3) group providing a nontrivial metric for the scalar product of two SU(2) operators.

The primary direction to this work came from the observation, that the continuum $1/r^2$ model on a ring, i.e., the Sutherland model [4], with a Hamiltonian

$$H=\sum p_i^2+\phi^2\lambda(\lambda-1){\sum}'\sin^{-2}\phi(x_i-x_j)-\text{const}\,,$$

can be factorized [5] in the sense of Darboux [6], and written as $\sum Q_n^\dagger Q_n$, with $Q_n=p_n+i(\lambda\phi)\sum_m'\cot\phi(x_n-x_m)$. Further it is easy to see that the operators Q_n annihilate the Jastrow-Sutherland wave function $\prod\sin^\lambda\phi(x_n-x_m)$, whereby one has the rigorous result that the wave function is the ground state of the stated model. Similar results hold [5] for the Calogero [7] system as well, and the existence of annihilators is trivial to establish for an arbitrary Jastrow function. The operator $\xi=\sum Q_n C_n^\dagger$, where C_n is a fermionic operator thus provides a basis for a supersymmetric description of the model, and this factorization seems to be important in elucidating the algebraic structure that supports this family of exactly solvable models [5]. It is thus natural to ask if a similar factorization holds for the discrete $1/r^2$ model.

In this work, I first construct an "inner product" between, or more precisely a bilinear form of, two SU(2) operators in the fundamental representation of the SU(2) group (i.e., $|S|=\frac{1}{2}$), using a "metric" picked from SU(3), and show that this leads to operators that are effectively the square roots of H_0. The resulting sets of operators are then shown to annihilate Ψ_g.

We begin by defining dual variables $\psi_i^a=\sum_{j\ne i}\chi_{i,j}S_j^a$. Here $\chi_{i,j}$ is assumed to be an odd (real) function of x_i-x_j; it is in fact $\phi\cot\phi(x_i-x_j)$ for the model considered in detail, but more general forms are possible, and we impose conditions on it below. Further, I define a scalar product [8]

$$V_i[\lambda]=S_i^a g_{a,\beta}(\lambda)\psi_i^\beta\,. \tag{2}$$

When the "metric tensor" g is the unit matrix, the operator V reduces, apart from a term $i\mathbf{S}_i\cdot\mathbf{S}_{\text{tot}}$ to a mutually commuting set of operators considered by Gaudin [9], as the small coupling limit of the Yang Y operators [10], and similar operators have been recently discussed by Laughlin [11]. My generalization involves picking the metric g from the eight SU(3) generators, i.e., the Hermitian traceless matrices. Recall that the Gell-Mann matrices [12] for $\lambda=1, 3, 4, 6, 8$ are real symmetric which we denote collectively by $\lambda\in\mathcal{S}$, and the rest, $\lambda=2, 5, 7$, are imaginary antisymmetric (denoted by $\lambda\in\mathcal{A}$). We square the above to find

$$V_i[\lambda]^\dagger V_i[\lambda]=\tfrac{1}{8}\sum_j{}'\chi_{i,j}^2+\tfrac{1}{4}\sum_{j,k}{}'S_j^aS_k^a\chi_{i,j}\chi_{i,k}M[\lambda|a,a] -\tfrac{1}{4}\sum_j{}'S_i^aS_j^a\chi_{i,j}^2F[\lambda|a,a]\,, \tag{3}$$

where we use the convention throughout that a prime on the sums indicates that all indices in the summand are distinct, with $M[\lambda|\alpha\beta]=(g(\lambda)^\dagger g(\lambda))_{\alpha,\beta}$, ϵ the antisymmetric tensor, and

$$F[\lambda|\alpha,\beta]=\epsilon(\rho,\sigma,\alpha)\epsilon(\gamma,\delta,\beta)g^\dagger_{\rho,\gamma}(\lambda)g_{\sigma,\delta}(\lambda)\,.$$

The tensor F is easy to evaluate on writting a SU(3) representation for the ϵ symbol $(W^\alpha)_{\beta,\gamma}=\epsilon(\alpha,\beta,\gamma)$ giving $F[\lambda|\alpha,\beta]=-\text{Tr}[W^\alpha\cdot g\cdot W^\beta\cdot g]$. It is readily seen that both M and F are diagonal tensors, leading to the result that the right-hand side (RHS) of Eq. (3) is diagonal in

TABLE I. The tensors M and F [$e_1=\{1,0,0\}_{\text{diag}}$, $e_2=\{0,1,0\}_{\text{diag}}$, $e_3=\{0,0,1\}_{\text{diag}}$].

λ	1	2	3	4	5	6	7	8
$M[\lambda]$	e_1+e_2	e_1+e_2	e_1+e_2	e_1+e_3	e_1+e_3	e_2+e_3	e_2+e_3	$1/3(e_1+e_2+4e_3)$
$F[\lambda]$	$-2e_3$	$2e_3$	$-2e_3$	$-2e_2$	$2e_2$	$-2e_1$	$2e_1$	$1/3(-4e_1-4e_2+2e_3)$

the spin space for all λ. The tensors M and F are given in Table I. We next impose a constraint on the functions χ so that the sum over i in Eq. (3) leads to interesting results. We require χ to satisfy a functional equation with $u=\phi(x_i-x_j)$, $v=\phi(x_j-x_k)$, $w=\phi(x_k-x_i)$, $u+v+w=0$,

$$\chi(u)\chi(v)+\chi(u)\chi(w)+\chi(v)\chi(w)=f(u)+f(v)+f(w)+\psi(u)\psi(v)\psi(w)\,. \tag{4}$$

This equation expresses a separation of variables of a kind familiar from Jacobian elliptic functions and has three explicit realizations that I am aware of [13], with a triad of functions $\{\chi||f||\psi\}$: (a) $\{\phi\cot||\phi^2/3||0\}$; (b) $\{\theta_1'/\theta_1||(-\eta_1 L)/\pi^2-(1/2)\theta_1''/\theta_1||0\}$; and (c)

$$\{\mathrm{cn}/\mathrm{sn}||1/3(1+m/4)+(1/2)(\mathrm{dn}-1)/\mathrm{sn}^2||1/(2m)^{2/3}[(1-\mathrm{dn})/\mathrm{sn}]^2\}\,.$$

Case (a) is of primary importance in this work, and the new case (c) is of interest in the continuum problem, since by an elementary extension of our arguments [5], the Jastrow wave function $\prod\{1-\mathrm{dn}[\Lambda(x_i-x_j)]/\mathrm{sn}[(x_i-x_j)\Lambda]\}^\lambda$ is the exact ground state of a model of many particles with $H=\sum Q_n^\dagger Q_n$, and $Q_n=p_n+ig\sum \mathrm{cn}/\,\mathrm{sn}[\Lambda(x_i-x_j)]$, corresponding to two-body and three-body interactions of a particular type, which generalizes and contains the Sutherland $1/\sin^2$ problem as a special case. We will return to its detailed study in a later work, we merely note here that the three-body term in the continuum problem, in fact *reduces to a two-body term in the discrete case.* The sum over i can be carried out in Eq. (3) and using Eq. (4) we find

$$\sum_i V_i^\dagger[\lambda]V_i[\lambda]=\frac{L}{8}\chi_0+\tfrac{1}{2}\sum_{i<j}{}' S_i^a S_j^a(2M[\lambda|a,a]-F[\lambda|a,a])\chi_{i,j}^2+M[\lambda|a,a]\{(4-L)f_{i,j}-\tilde{\psi}_{i,j}-f_0\} \tag{5}$$

with $\chi_0=\sum'\chi_{i,j}^2$, $f_0=\sum' f_{i,j}$, and $\tilde{\psi}_{i,j}=\sum'\psi_{i,j,k}$. We now observe that the combination $2M-F$ appears in this function, and we give it a name $K[\lambda|a,a]=2M-F$. Henceforth we specialize to case (a) for the function χ, i.e., the trigonometric case, and find

$$\sum_i V_i^\dagger[\lambda]V_i[\lambda]=L(L-1)(L-2)\frac{\phi^2}{24}+\frac{\phi^2}{4}(LM[\lambda|a,a]-F[\lambda|a,a])[L/4-(S_t^a)^2]$$
$$+\frac{\phi^2}{2}\sum_{i<j}{}' S_i^a S_j^a K[\lambda|a,a]/\sin^2\phi(x_i-x_j)\,, \tag{6}$$

where S_t^a is the total spin operator. We now consider the case $\lambda\in\mathcal{S}$, wherein $K=2\{1,1,1\}_{\text{diag}}$ [8]. Hence for this class we find (with Hermitian V's)

$$\sum_i (V_i[\lambda])^2=H_0-E_0-\frac{\phi^2}{4}(S_t^a)^2(LM[\lambda|a,a]-F[\lambda,a,a])\,. \tag{7}$$

Note that the nontrivial third term in the RHS of Eq. (6), manages to be a rotational scalar owing to a delicate interplay between the structure of the functional Eq. (4) (giving the factor of 2 in front of M in K), and of the structures of F and K.

We next turn to the case $\lambda\in\mathcal{A}$. The structure of the function K suggests that we define a triad $L_i^x=iV_i[7]$, $L_i^y=-iV_i[5]$, and $L_i^z=iV_i[2]$, these are "angular-momentum-like," in the sense that we can express $L_i^a=\sum\epsilon(a,\beta,\gamma)\chi_{i,j}S_i^\beta S_j^\gamma$. We further need two results that can be obtained by explicit summations. The dual variables ψ_i^a satisfy [8]

$$\phi^{-2}\sum_i(\psi_i^a)^2=4\sum_{i<j}S_i^a S_j^a/\sin^2\phi(x_i-x_j)-L(S_t^a)^2+L(L^2+2)/12 \tag{8}$$

and

$$(-i)\phi^{-2}\sum_i[L_i^a,\psi_i^a]=2\sum_{i<j}S_i^\gamma S_j^\gamma/\sin^2\phi(x_i-x_j)+L/2-\sum_\gamma(1-\delta_{a,\gamma})(S_t^\gamma)^2\,. \tag{9}$$

Putting these together, we find for $a=x,y,z$,

$$\sum_i(L_i^a+i\psi_i^a)(L_i^a-i\psi_i^a)=3(H_0-E_0)-(\phi^2/4)\sum_\gamma(S_t^\gamma)^2[L+4+3(L-2)\delta_{a,\gamma}]\,. \tag{10}$$

We next discuss the action of the various operators defined above, and propose to prove for all $i \in [1,L]$, that $V_i[\lambda \in \mathcal{S}]|\Psi_g\rangle = 0$ and further that $(L_i^a - i\psi_i^a)|\Psi_g\rangle = 0$ for $a = x,y,z$. Towards this end we first state two results $\mathcal{Q},\mathcal{R}$, and assuming these, prove the annihilation of $|\Psi_g\rangle$ by these operators. Later we prove these two results, completing the program. The results in question are as follows: For $i \in [1,L]$,

$$\mathcal{Q}\text{: } \{Q_i|\Psi_g\rangle = 0 | Q_i \equiv \sum_j{}' \chi_{i,j}[S_i^+S_j^- - 2(S_i^z + \tfrac{1}{2})(S_j^z + \tfrac{1}{2})]\} \tag{11}$$

and

$$\mathcal{R}\text{: } \{R_i|\Psi_g\rangle = 0 | R_i \equiv \sum_j{}' \chi_{i,j} S_i^- S_j^-\} , \tag{12}$$

where $S_i^+ = S_i^x + iS_i^y$, etc., and $|\Psi_g\rangle$ is written in the standard S^z basis [1] as $\sum \Psi_g \prod S_i^+ |\downarrow \cdots \downarrow\rangle$. It is useful to display the structure of $V[\lambda]$. Recall

$$V_i[8] = 1/\sqrt{3} \sum_j{}' \chi_{i,j}[S_i^xS_j^x + S_i^yS_j^y - 2S_i^zS_j^z] \tag{13}$$

which we denote $V[8] = 1/\sqrt{3}[xx + yy - (2)zz]$, and in this abbreviated notation, $V[1] = [xy + yx]$, $V[3] = [xx - yy]$, $V[4] = [xz + zx]$, $V[6] = [yz + zy]$, and of course $L^a = \sum_{\beta,\gamma} \epsilon(a,\beta,\gamma)[\beta,\gamma]$. We first consider the operation of $U = \prod S_i^x$ on $\mathcal{Q}$, this is a particle-hole transformation operator, and sends $(S_i^+, S_i^-, S_i^z) \rightarrow (S_i^-, S_i^+, -S_i^z)$, and gives an independent result, which we add and subtract to the original one (we use $\sum_j' \chi_{i,j} = 0$ for L an even integer), giving $V_i[8]|\Psi_g\rangle = 0$, and also $(L_i^z - i\psi_i^z)|\Psi_g\rangle = 0$. Using the fact that $|\Psi_g\rangle$ is a singlet, this result is sufficient to show that $L_i^a - i\psi_i^a$ are annihilators, for all a. We next consider the result $\mathcal{R}$, and its partner under U, adding and subtracting we find $V_i[1]|\Psi_g\rangle = 0$ and $V_i[3]|\Psi_g\rangle = 0$, and using rotational invariance of Ψ_g, conclude that $V_i[4]$ and also $V_i[6]$ are annihilators.

We next prove the two results $\mathcal{R},\mathcal{S}$. For this purpose, it is useful to absorb the "Marshall" phase factor of Ψ_g into the operators (through a unitary $U' = \prod S_i^z$), and define an oscillating function $\tilde{\chi}_{i,j} = (-1)^{x_i - x_j}\chi_{i,j}$, whereby the transverse terms of Q_i, R_i are redefined with the $\tilde{\chi}_{i,j}$ function. Consider first the operation of Q_i on $|\Psi_g\rangle$; we commute the S_i^- to the right, where it annihilates $|\downarrow \cdots \downarrow\rangle$, hence we collect as a common factor (in the tilde representation)

$$\sum_j \tilde{\chi}_{i,j}\Psi_g(x_1,\ldots,x_j,\ldots,x_{L/2}) - \Psi_g(x_1,\ldots,x_{j-1},x_i,x_{j+1},\ldots)\left(2\sum_{k\in[1,L/2]}{}' \chi_{i,k}\right) . \tag{14}$$

We now use the elementary identity for $q = (2\pi/L) \times \text{int}$, and $|q| < \pi$,

$$\sum_{n=1}^{L-1} (-1)^n \cot\left(\frac{n\pi}{L}\right) \exp(-inq) = i\frac{Lq}{\pi} , \tag{15}$$

and the result vanishing for $|q| = \pi$. This implies that the cotangent function acts as a linear derivative on plane waves inside the first (Brillouin) zone (FBZ). Further, we note that under multiplication, generic periodic functions on $[1,L]$ possessing arbitrary momenta (i.e., Fourier components), require "umklapp" reduction (i.e., reducing momenta mod2π) to reduce the product function into the FBZ. "Good functions" can now be defined as the *class of functions which can be expressed as products of periodic functions that do not require umklapp reduction.* Typical examples are products of plane waves with small enough momenta, and in fact Ψ_g was shown [1] to be a good function in precisely this sense. The property of convolution with $\sin^{-2}$ in place of cot in Eqs. (14) and (15) was shown to be analogous to the second derivative. For good functions, it is possible to show the existence of the Liebnitz product rule, and the convolution with cot above is a bona-fide first derivative for functions f in this class, i.e., $\sum_j \tilde{\chi}_{i,j} f(x_j) = f'(x_i)$. This immediately implies that Eq. (14) vanishes identically, proving $\mathcal{Q}$.

In order to prove $\mathcal{R}$, we push the $S_i^- S_j^-$ operator to the right and collect a common term

$$\sum_j (-1)^{x_i - x_j} \chi(\phi(x_i - x_j))\xi(x_i - x_j) \prod_{k=1}^{L/2-2} \xi(x_j - x_k) , \tag{16}$$

where $\xi(x) \equiv \sin^2[\pi/L(x_i - x_j)]$, and the set of variables x_k; $k = 1,\ldots,L/2-2$, stands for arbitrary locations of the $L/2-2$ "spectator particles." I now claim that this vanishes identically by using the elementary identity

$$\sum_{n=1}^{L-1} \cot(n\phi)\sin^2(n\phi)\exp(iqn) = 0 , \tag{17}$$

unless $|q| = 2\pi/L$. This is easy to see by expanding the product term in Eq. (16), and examining the total momentum involved. This completes the proof of the result $\mathcal{R}$.

It is easy to see from Eqs. (10) and (7), that the following rotationally invariant representations follow for H_0:

$$H_0 = E_0 + \tfrac{1}{9} \sum_a \sum_i (L_i^a + i\psi_i^a)(L_i^a - i\psi_i^a) + \tfrac{3}{2}(L+1)\phi^2(S_t)^2 , \tag{18}$$

and

$$H_0 = E_0 + \tfrac{1}{5} \sum_{\lambda \in \mathcal{S}} \sum_i (V_i[\lambda])^2 + \tfrac{1}{6}(L+1)\phi^2(S_t)^2 . \tag{19}$$

It is now obvious that E_0 is the ground-state energy of H_0, we take, say Eq. (19), the right-hand side consists of positive terms that annihilate Ψ_g. We may take any positive polynomial in $V_i[\lambda]^2$, and this operator also has Ψ_g as its ground state. A general operator of this kind is not rotationally invariant, in fact none of the V's are, while Ψ_g is. We have thus another interesting example of a situation, where the "invariance of the vacuum exceeds that of the world" [14,15].

The operator algebra of V's is highly nontrivial, and is currently under study. We may regard $\eta = \sum V_i C_i^\dagger$, with C_i a fermion, as giving rise to the square root of H_0, in the sense that it is the bosonic part of $\{\eta, \eta^\dagger\}$. The functional equations (16) and (4) seem to be promising for further study, in order to obtain other soluble models. It is interesting that the discrete model may be regarded as a second quantization of the continuum model, since the different magnetization sectors correspond to different particle numbers. This point of view provides some novel insights into the nature of this class of states. The property $\mathcal{R}$, for instance (which is clearly valid at any particle number $\leq L/2$), is very hard to see in the first quantized problem, and expresses a fundamental incompressibility of these Jastrow states.

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