Nonequilibrium Dynamics of Driven Line Liquids

Terence Hwa
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 14 May 1992)

We study the nonequilibrium dynamics of line liquids as in the nonlinear motion of flux lines of a superconductor driven by an applied electric current. Our analysis suggests a transition in the dynamics of the lines from a smooth, laminar phase at small driving forces, to a rough, turbulent phase when the drive is increased. We explore the nature of these phases and describe interesting analogies to driven diffusion and growing interfaces.

PACS numbers: 64.60.Ht, 05.40.+j, 05.60.+w, 74.60.Ge

The statistical mechanics and dynamics of extended objects such as lines have attracted increasing attention, especially in the context of flux lines of a superconductor [1-4], in recent years. In this paper, we study the *nonequilibrium* dynamics of a driven line liquid. Our analysis suggests that such a system can exhibit a transition from a "laminar" phase described by equilibrium fluctuations at small driving forces to a "turbulent" phase dominated by nonlinear effects as the driving force is increased. We explore the nature of these phases and describe interesting analogies to driven diffusion [5] and growing interfaces [6].

As is well known for type-II superconductors [7], magnetic flux lines can penetrate a sample under the application of an external magnetic field **H**, forming an Abrikosov lattice at low temperatures. However, as pointed out by Nelson [1], thermal fluctuations can destroy the translational order of the flux lines, melting the flux lattice into a line liquid. The equilibrium properties of the line liquid has been studied in detail by Nelson and coworkers [1,3,4]. This paper concerns the *nonequilibrium* dynamics of the flux line liquid driven by an applied electric current $\mathbf{J}_e \perp \mathbf{H}$. But before delving into the driven dynamics, we shall first give a brief review of the dynamics of the lines in equilibrium.

We describe the long-wavelength, long-time behavior of the line liquid by a vector field $\mathbf{B}(\mathbf{r},t)$, with the constraint $\nabla \cdot \mathbf{B}=0$. The divergentless **B** field lends itself naturally to the description of a generic collection of lines without free ends. In the context of superconductors, **B** is the coarse-grained magnetic field carried by the flux lines. Let the external field **H** be in the z direction, then the z component of **B** describes the density of the flux lines and the x,y (or collectively called the $\perp$) components describe the local tilt [4]. The dissipative thermal motion of the flux lines in the liquid phase destroys the superconductivity of the sample [8]. The resulting dynamics is diffusive and can be described by the Langevin equation [4], $\partial_t \mathbf{B} = \nu \nabla^2 \mathbf{B} + \nabla \times \boldsymbol{\eta}$, where we have scaled out the trivial anisotropy factors between the z and $\perp$ directions. The Langevin noise is employed to summarize thermal fluctuations over the microscopic degrees of freedom. It must take on the form of a curl to keep **B** divergentless, but is otherwise taken to be uncorrelated and Gaussian distributed, with the second moment $\langle \eta_i(\mathbf{r},t)\eta_j(0,0)\rangle = 2D\delta_{ij}\delta^3(\mathbf{r})\delta(t)$. The choice of the noise spectrum is justified by matching the resulting static structure factor, $\langle \mathbf{B}(\mathbf{k},t)\mathbf{B}(-\mathbf{k},t)\rangle$, with the same quantity obtained from the equilibrium theory [3]. The phenomenological parameters D and ν can be identified as the temperature and elastic moduli of the equilibrium theory.

We now consider nonequilibrium effects introduced by an applied external current $\mathbf{J}_e \perp \hat{\mathbf{z}}$, taken here to be in the y direction. The external current provides a Lorentz force $\mathbf{F}=\mathbf{J}_e \times \mathbf{B}$ which drives the line liquid in the x direction. The viscous motion [4] of the flux lines in turn induces an electric field **E** which must be included in the equation of motion, i.e., $\partial_t \mathbf{B} \rightarrow \partial_t \mathbf{B} - \nabla \times \mathbf{E}$. The most general form of **E** allowed by symmetry is

$$\mathbf{E} = \rho(B)\mathbf{J}_e + \tilde{\rho}(B)(\hat{\mathbf{B}}\cdot\mathbf{J}_e)\hat{\mathbf{B}}, \tag{1}$$

where B and $\hat{\mathbf{B}}$ denote the norm and direction of **B**, respectively, and the coefficients $\rho,\tilde{\rho}$ can be identified as the field-dependent resistivity [9].

For simplicity, we make a reasonable assumption that the fluctuation of **B** in the direction of $\mathbf{J}_e$ is small (i.e., $B_y \ll B_x, B_z$), and only consider the effect of the first term in Eq. (1). We will later show the self-consistency of this assumption. The dynamics is then more conveniently expressed in terms of the vector potential **A**. With $\mathbf{B}=\nabla\times\mathbf{A}$, the equation of motion becomes

$$\frac{\partial \mathbf{A}}{\partial t} = \nu\nabla^2\mathbf{A} - \rho(B)\mathbf{J}_e + \boldsymbol{\eta} + \nabla\phi, \tag{2}$$

where the last term will be fixed by the choice of gauge. We look for the fluctuation of **A** about the average by using a *displacement* field, $\mathbf{u} = \hat{\mathbf{z}} \times (\mathbf{A} - B_0 x \hat{\mathbf{y}})$, where B_0 is the mean density of the lines. Expanding $\rho(B)$ and keeping the leading order nonlinearity, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} = \nu\nabla^2\mathbf{u} + \hat{\mathbf{T}}\left[E_0 - v(\nabla_\perp\cdot\mathbf{u}) + \frac{\lambda_\perp}{2}(\nabla_\perp\cdot\mathbf{u})^2 + \frac{\lambda_z}{2}\left(\frac{\partial \mathbf{u}}{\partial z}\right)^2\right] + \boldsymbol{\xi}. \tag{3}$$

In Eq. (3), $\hat{\mathbf{T}}=\hat{\mathbf{J}}_e\times\hat{\mathbf{z}}=\hat{\mathbf{x}}$ is the transport direction, $-\nabla_\perp\cdot\mathbf{u}=B_z-B_0$ is the density fluctuation, and $\partial_z\mathbf{u}=\mathbf{B}_\perp$ describes the local tilt (see Refs. [3] and [4]). The parameters in square brackets are given by the Taylor expansion of $\rho(B)$ about B_0. For example, $E_0=\rho(B_0)J_e$ is the mean electric field induced; it produces an overall translation in $u_x\equiv\mathbf{u}\cdot\hat{\mathbf{T}}$ and can be shifted away. Similarly, $v=\rho'(B_0)J_e$ is the mean drift rate of the lines; it is again shifted away by a Lorentz boost in the $\hat{\mathbf{T}}$ direction. The nonlinear terms remaining are characterized by the coefficients $\lambda_\perp=\rho''(B_0)J_e$ and $\lambda_z=\rho'(B_0)J_e/B_0$. For superconductors, $\rho(B)$ is usually a monotonically increasing function, saturating at large values of B [10]. This implies that E_0, v, and λ_z are always positive, while $\lambda_\perp$ may take on either sign. While the response of the system to a sign change in $\lambda_\perp$ is an interesting study in itself [11], we shall be concerned with the situation $\lambda_\perp>0$ in this study. Anticipating the restoration of isotropy, we set $\lambda_\perp=\lambda_z=\lambda$ from here on. Finally, the new noise is $\boldsymbol{\xi}=\hat{\mathbf{z}}\times(\boldsymbol{\eta}+\nabla_\perp\phi)$, with $\partial_z\phi=\eta_z$ from the gauge choice $A_z=0$ in Eq. (2). This gives $\langle\xi_i(\mathbf{k},t)\xi_j(\mathbf{k}',t')\rangle=2D_{ij}(\mathbf{k})(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')\delta(t-t')$ in Fourier space, with the noise spectrum

$$D_{ij}(\mathbf{k})=D[\delta_{ij}+(k_\perp^2\delta_{ij}-k_\perp^ik_\perp^j)/k_z^2]\,. \tag{4}$$

It is worth emphasizing that although Eq. (3) is derived through some rather specific considerations such as the form of $\rho(B)$, the general form of the dynamics is a consequence of the applied driving force $\mathbf{J}_e$ which breaks spatial isotropy. As a result, we expect Eq. (3) to be a good description for a generic line liquid with dynamics local in $\mathbf{B}$ and $\mathbf{u}$.

A systematic investigation of the scaling behaviors is the method of dynamic renormalization group (DRG) [12]. In that approach, we generalize the space of $\perp$ directions (and thereby the components of the displacement field $\mathbf{u}$) from two to $d_\perp$ dimensions. Also we generalize the z direction from one to d_z dimensions. Equation (3) is studied near the critical dimension $d=d_\perp+d_z=2$, and the result is then analytically continued to the physical dimension $(d_\perp,d_z)=(2,1)$. The most important output of the DRG analysis is a recursion relation which describes the relevancy of the nonlinearity at successively larger length scales. In a one-loop study, the recursion relation has the form

$$\frac{dg}{dl}=(2-d)g+I(d_\perp,d_z)g^2\,, \tag{5}$$

where the coupling constant $g\sim\lambda^2D/v^3$ characterizes the effective strength of nonlinearity, $I(d_\perp,d_z)$ is the one-loop result evaluated *along* the line of critical dimension $d_\perp+d_z=2$, and $l\sim\ln(1/k)\to\infty$ is the infrared limit of interest.

In the physical dimension $d=3$, the first term in Eq. (5) is negative, indicating the irrelevancy of a small nonlinearity. However, the nonlinearity can become relevant if higher-order terms in the recursion relation are positive. For example, if $I(d=2)>0$, then a dynamic phase transition occurs at $g_c=(d-2)/I$. For $g<g_c$, the nonlinearity is still irrelevant. But for $g>g_c$, the coupling constant flows to large values and the system is described by a *strong coupling* fixed point [13]. Thus the behavior of the system at the physical dimension is dictated to a large extent by what happens right at the line of critical dimensions, i.e., by the sign of $I(d=2)$. To better understand the behavior of the system then, we shall now consider some limiting cases on the critical line.

We first examine the case $(d_\perp,d_z)=(2,0)$, corresponding to the situation where an external field $\mathbf{H}$ is applied normal to a thin-film superconductor and the external current is applied in the plane. In this case, the flux lines become point vortices and the problem becomes quite simple. Since $\partial_z\mathbf{u}=0$, the configuration of the vortices can be described by a scalar field $n=-\nabla_\perp\cdot\mathbf{u}$ which is the fluctuation of the vortex density. Equations (3) and (4) then reduce to the well-known driven-diffusion system (DDS) [5]. The DRG method has been used to investigate the scaling properties of that system. One finds $I(d_\perp=2,d_z=0)<0$, indicating the marginal irrelevancy of the nonlinearity.

We next investigate the limit $(d_\perp,d_z)=(1,1)$. This corresponds to applying an external field $\mathbf{H}$ in the plane of a thin-film superconductor, with an applied electric current normal to the plane as shown in Fig. 1(a). In this special configuration, the displacement field $\mathbf{u}$ is reduced to a scalar, u_x. The noise spectrum also simplifies as the nonlocal term vanishes. Equations (3) and (4) then become the simpler anisotropic Kardar-Parisi-Zhang (KPZ) equation which describes the growth of a tilted crystalline surface [6,11], with u_x being the height of the surface. The connection between the lines and interfaces is intuitively simple: The lines shown in Fig. 1(a) can be viewed as the contour plot of a tilted surface [14]. The growth of such a surface corresponds to the movement of the lines to the right.

For the KPZ equation, one finds the one-loop term in Eq. (5) to be positive [6]. So the nonlinearity is marginally *relevant*, and the asymptotic scaling behavior is de-

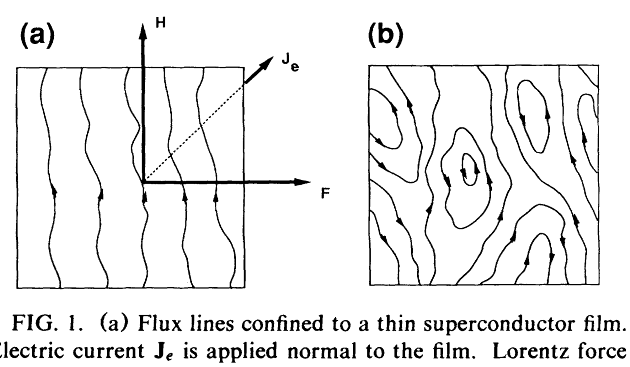

FIG. 1. (a) Flux lines confined to a thin superconductor film. Electric current $\mathbf{J}_e$ is applied normal to the film. Lorentz force $\mathbf{F}$ drives the lines to the right. (b) A "scrambled" line configuration from the contour plot of a *rough* surface.

scribed by a strong coupling fixed point. Extensive numerical studies [15] find that fluctuations in u_x diverge algebraically with system size in the strong coupling phase. In the interface language, this corresponds to the divergence of interfacial width, and the resulting surface is called "rough." A contour plot of a rough surface yields a "scrambled" line configuration with the proliferation of vortex loops [see Fig. 1(b)]. In contrast, a smooth interface gives a set of roughly parallel lines without loops [Fig. 1(a)].

The physical problem of a line liquid in three dimensions lies somewhere in between the KPZ and DDS limits. To find out whether this problem may also flow to strong coupling, we need to know the boundary between the KPZ- and DDS-dominated regions in the space of generalized dimensions $(d_\perp, d_z)$. We can get an estimate by computing the function I in Eq. (5) and see where it changes its sign. However, a full DRG treatment of Eq. (3) is rather cumbersome. Here we discuss a truncated version which contains the essential physics.

From Eq. (3), we note that u_x is the only component of $\mathbf{u}$ renormalized by the nonlinear terms. If the system flows to strong coupling, then only u_x will pick up anomalous scaling. It will then dominate the dynamics, allowing us to replace $\partial\mathbf{u}$ by ∂u_x in Eq. (3) in the hydrodynamic limit. [This is also the justification for neglecting the cross term in Eq. (1).] We also find the nonlocal part of the noise spectrum in Eq. (4) not to renormalize, while the local part of $\langle \xi_x \xi_x \rangle$ does. So the nonlocal part can again be ignored in the strong coupling limit. Consequently, isotropy between the x and z directions is restored. In the comoving frame $x \to x - vt$, $u_x \to E_0 t + h$, we obtain the following simpler equation of motion:

$$\frac{\partial h}{\partial t} = \nu[\nabla_x^2 + \nabla_z^2]h + \nu_y \nabla_y^2 h + \frac{\lambda}{2}[(\nabla_x h)^2 + (\nabla_z h)^2] + \eta\,, \tag{6}$$

where $\eta = \xi_x$ is now a white noise. We believe the above system is equivalent to Eqs. (3) and (4) in the strong coupling limit. Note that Eq. (6) is different from the KPZ equation [6] by a "missing " term $(\nabla_y h)^2$. The exclusion of such a term is a consequence of the *locality* of the dynamics in $\mathbf{B}$. This missing term gives rise to anisotropy between the (x,z) and y directions, which we explicitly take into account by allowing the diffusion coefficients in the two directions to be different.

Equation (6) is straightforwardly analyzed by the DRG method. We find the one-loop term in the recursion relation (5) to be $I(d_\perp, d_z) \sim d_z(19 - 3d_\perp) - (16 - 9d_\perp + d_\perp^2)$, where $d_\perp = 1 + d_y$. It changes sign at $d_z^* \approx 0.24$ along the critical line $d=2$. For $1 \geq d_z > d_z^*$, we have $I(d=2) > 0$, and the system flows to strong coupling as in the KPZ limit $(d_z = 1)$. However, for $d_z < d_z^*$, the nonlinearity is irrelevant. There, the series of approximations leading to Eq. (6) are no longer valid as various components of $\mathbf{u}$ become comparable. In fact, in the limit $d_z = 0$, Eq. (6) becomes the DDS with anisotropic noise [5], which actually belongs to a universality class different from the $d_z = 0$ limit of Eq. (3) (DDS with isotropic noise). Nevertheless, we expect the result for $d_z > d_z^*$ to be good, and find that the KPZ-like strong coupling behavior dominates for $\sim \frac{3}{4}$ of the way along the critical line.

To obtain the boundary of the strong coupling region away from the critical line $d=2$, we will need to carry out the DRG calculation to higher order. However, we may use the full one-loop result to get a "feel" of the boundary in the vicinity of the critical line. Solving for the root of $I(d_\perp^*, d_z^*) = 0$, we obtain a tentative boundary which is sketched in the space of generalized dimensions $(d_\perp, d_z)$ to guide the eye (see Fig. 2). The result suggests that KPZ-like behavior is likely to dominate a large portion of the $(d_\perp, d_z)$ space. It is then reasonable to expect the existence of a phase transition to strong coupling in the physical dimension $(d_\perp, d_z) = (2,1)$. However, numerical simulations of Eq. (6) in the physical dimension are needed to make a definitive conclusion.

In the remaining part of this paper, we shall assume that a phase transition for the driven line liquid does exist in three dimensions and explore its consequences. Pictorially, the transition is between a line configuration described by the three-dimensional generalization of Fig. 1(a) at small driving force J_e, to the one described by the generalization of Fig. 1(b) when J_e is increased beyond a critical value J_e^c. [The actual value $J_e^c \sim \nu^{3/2}/D^{1/2}\rho''(B_0)$ will depend on the specifics of material and bias conditions.] If we view the lines as the streamlines of a fluid flow, then the two phases correspond to laminar and turbulent flows, respectively. The laminar phase is described by the linear theory [4]. However, the turbulent phase is much more complicated. From experience [6,15] with the KPZ equation in three dimensions $(d_\perp, d_z) = (1,2)$, we expect both the renormalized diffusion coefficient and noise amplitude to diverge algebraically in the infrared

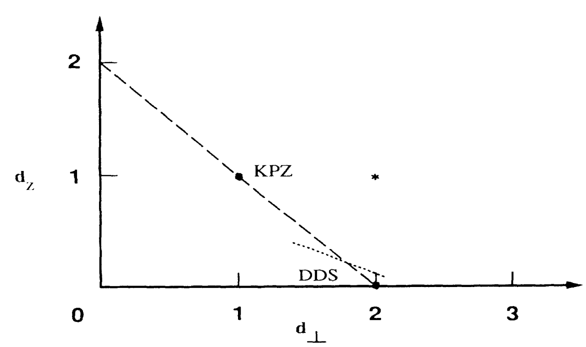

FIG. 2. The space of generalized dimensions $(d_\perp, d_z)$. The dashed line is the line of critical dimensions $d_\perp + d_z = 2$. The dotted line is a tentative boundary separating the KPZ-like and the DDS-like dynamics (see text). The physical dimension of interest is marked by the asterisk.

limit $k \to 0$. These anomalies will lead to a singular structure factor. They will also give anomalous dynamics—the response of the system will spread faster than $\sqrt{t}$ for diffusion. Critical properties associated with the transition itself are also of interest. The nature of this type of phase transitions has been discussed in detail elsewhere [13]. Here we merely point out that associated with a diverging length scale at the critical point, we expect singularities in global quantities such as the renormalized E_0 in Eq. (3), i.e., $E_0(\text{sing}) \sim (J_e - J_e^c)^{2-\alpha}$. Such singularities should be detectable from simple I-V measurements.

As mentioned at the beginning of this paper, we expect the qualitative features of the driven dynamics discussed here to be applicable to a generic line liquid. It will be interesting to reexamine the proliferation of vortex loops in experiments of driven superfluid helium [16] in light of the phase transition discussed here. In addition to line liquids, the dynamics [Eq. (6) in particular] explored here provides a convenient link between the KPZ and DDS dynamics, which are two of the simplest generalization of the diffusion equation, and have appeared in a wide variety of nonequilibrium problems. This link may be exploited to obtain a perturbative access to the strong coupling fixed point itself.

The author would like to thank D. S. Fisher, M. Kardar, D. R. Nelson, and J. P. Sethna for useful discussions. This work was supported by NSF through Grant No. DMR-91-06237 and by IBM.

[1] D. R. Nelson, Phys. Rev. Lett. **60**, 1415 (1988); D. R. Nelson and S. Seung, Phys. Rev. B **39**, 9153 (1989).
[2] M. P. A Fisher, Phys. Rev. Lett. **62**, 1415 (1989); D. S. Fisher, M. P. A. Fisher and D. A. Huse, Phys. Rev. B **43**, 130 (1991).
[3] D. R. Nelson and P. Le Doussal, Phys. Rev. B **42**, 10113 (1990).
[4] M. C. Marchetti and D. R. Nelson, Physica (Amsterdam) **174C**, 40 (1991).
[5] H. K. Janssen and B. Schmittman, Z. Phys. B **63**, 517 (1986).
[6] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang, Phys. Rev. A **39**, 3053 (1989).
[7] M. Tinkham, *Introduction to Superconductivity* (Krieger, Malabar, 1975).
[8] Point defects, which preserve superconductivity at low temperatures by collectively pinning the flux lines [2], are irrelevant in the liquid phase [3] and will be neglected in this study.
[9] For layered cuprate superconductors, the resistivity also depends on the angle between the applied field and the *c* axis. We shall not include this complication here as it will not change the qualitative result of our study.
[10] Y. B. Kim, C. F. Hempstead, and A. R. Strand, Phys. Rev. **139**, A1163 (1965); Y. Iye, in *Proceedings of the IBM-Japan International Symposium on Strong Correlation and Superconductivity, Mt. Fuji, Japan, May 1989*, edited by H. Fukuyama, S. Maekawa, and A. P. Malozemoff, Springer Series in Solid State Sciences Vol. 89 (Springer-Verlag, Berlin, 1989).
[11] D. Wolf, Phys. Rev. Lett. **67**, 1783 (1991).
[12] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977), and references therein.
[13] T. Hwa, Harvard University report, 1991 (to be published).
[14] The duality between the "lines" and "interfaces" was discussed in F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981). This was used to advance the theory of vortex glass in Ref. [2].
[15] For recent reviews, see T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989); J. Krug and H. Spohn, in *Solids Far From Equilibrium: Growth, Morphology and Defects,* edited by C. Godreche (Cambridge Univ. Press, New York, 1990).
[16] K. W. Schwarz, Phys. Rev. Lett. **64**, 1130 (1990), and references therein.