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Chaotic Billiards Generated by Arithmetic Groups

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It is known that statistical properties of the energy levels for various billiards on a constant-negative-curvature surface do not follow the universal random-matrix predictions. We show that nongeneric behavior of the systems investigated so far originates from the special arithmetic nature of their tiling groups, which produces an exponentially large degeneracy of lengths of periodic orbits. A semiclassical study of the two-point correlation function shows that the spectral fluctuations are close to Poisson-like ones, typical of integrable systems.

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The spectral fluctuation properties are now largely trusted as significant tests of underlying classical chaos in quantum dynamics. It was conjectured in [1] that for classically "enough chaotic" systems the energy levels should be distributed according to the predictions of the canonical random-matrix ensembles (RME) [2–4] characterized by level repulsion and long-range rigidity. This "universality conjecture" was checked later for many conservative systems and seems to be well confirmed (see, e.g., [3]). For time-reversal-invariant systems the expected ensemble is the Gaussian orthogonal ensemble (GOE), whereas for classically integrable systems the pattern of energy levels locally mimics a Poisson distribution [5].

Dynamical systems defined by geodesic motion on a compact surface of negative curvature are historical [6] and mathematical paradigms of classical chaos: Indeed, the induced Riemannian metric generates ergodicity, hyperbolicity, exponential decay of correlations, etc. (see e.g., [7]). We therefore expected to observe the universal RME fluctuation properties for their quantum analogs. The first large-scale computation of quantum spectra for polygonal billiards tiling the hyperbolic plane under a group of reflections surprisingly exhibited fluctuations closer to those of a Poisson spectrum than to a GOE one [8], whereas nontiling billiards behave according to the universality conjecture. Further numerical studies for various other systems tessellating the pseudosphere seem to confirm this "anomalous" behavior [8–13].

The main purpose of this Letter is to clarify the situation by studying a very special subclass of hyperbolic systems, namely, triangle billiards tiling under the action of so-called "arithmetic groups." The relevance of these groups stems from the fact that all tiling triangles considered so far, which display the quantum nongenericity, belong to this class. Arithmetic groups behave among all discrete groups more or less as integers among rational numbers. The unusual features of the spectrum are linked with the number-theoretical properties of these groups. Indeed it will be shown here that dynamics generated by such groups exhibit exponentially large degeneracy of the lengths of periodic orbits. It is the cumulative interference of periodic orbits with the same length which leads to nongeneric energy-level statistics.

On the contrary, an example of a tiling but nonarithmetic triangular billiard shows a good agreement with GOE statistics.

A necessary and sufficient condition for a triangle to tessellate the hyperbolic space under reflections on its sides is that its vertex angles should be equal to π/n, where n is any integer [14]. In the following, any such tiling triangle with angles $(\pi/m,\pi/n,\pi/p)$ will be denoted by (m,n,p). Three tiling triangles were considered in [8,10,11]: (i) (2,3,8), which is obtained by complete desymmetrization of the regular octagon [7]; (ii) $(2,3,\infty)$, which is half the modular domain (see, e.g., [14]); and (iii) (2,4,6), which was supposed to be taken at "random."

About 1700 quantum levels (with Dirichlet boundary conditions) were computed for each of those three billiards. In all cases, a considerable deviation of the spectral fluctuations from the GOE predictions was found.

As an illustration, Fig. 1 (taken from [11]) displays the Δ_3 statistics (a measure of the degree of spectral rigidity [2-4]) and the integrated nearest-neighbor spacing distribution for the triangle $(2,3,\infty)$. As the energy is increased, the results approach more and more the Poisson curve, i.e., strongly depart from GOE. Similar results were obtained in [9] for the spectrum of the Laplace-Beltrami operator on a genus-2 surface corresponding to functions invariant under the action of the regular octagon group.

On the contrary, the expected universal behavior was observed for nontiling billiards. For instance, the spectral properties of two triangles with angles $(\pi/2,17\pi/50,0)$ and $(\pi/2,\pi/8,67\pi/200)$ [10] were considered and found to be in good agreement with the generic GOE behavior [15].

These results emphasize the quantum nongenericity of the quoted tiling billiards, associated, as will be shown now, with very special discrete groups, the so-called arithmetic groups. The fact that at least some of these groups could have quantum and classical peculiarities has been known for a long time [16] but has not attracted wide attention.

Arithmetic groups are in some sense a generalization of the modular group, where instead of the (usual) integers there appear the integers of a certain (real) algebraic field. As precise definitions are quite tedious we shall omit them here, giving only main ideas. Details can be found, e.g., in [17].

Algebraic fields are constructed by adding to the field of ordinary rational numbers (denoted by Q) a root u of a given polynomial of finite degree:

$$u^n+a_{n-1}u^{n-1}+\cdots+a_0=0\,, \tag{1}$$

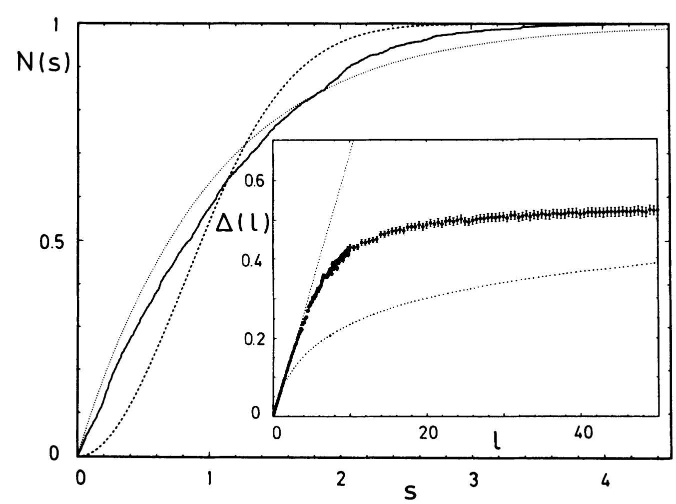

FIG. 1. The integrated nearest-neighbor distribution for 1700 energy levels of the triangle $(2,3,\infty)$; the dashed curve corresponds to the GOE distribution, and the dotted one to the Poisson. Inset: The Δ_3 statistics for the same triangle; the straight line corresponds to Poisson, and the curved one to GOE.

where all coefficients belong to Q. Every element of the field can be written as a linear combination of all powers of u up to $n-1$ with coefficients in Q:

$$z=m_0+m_1u+\cdots+m_{n-1}u^{n-1}\,. \tag{2}$$

In what follows we shall consider totally real fields where all roots of Eq. (1) are real and different. The substitution of another root u_j of this equation instead of u is called an automorphism of the field and will play an important role below. The integers of this field are defined in a similar way but the coefficients m_k should be such that z obeys a polynomial equation with integer coefficients, the dominant coefficient being 1 as in Eq. (1). In simple cases all m_k are the usual integers (or proportional to them).

Having defined the real field one can construct the so-called quaternion division algebra by considering matrices of the form [18]

$$\begin{pmatrix} x_1+x_2\sqrt{a} & (x_3+x_4\sqrt{a})\sqrt{b} \\ \sqrt{b}(x_3-x_4\sqrt{a}) & x_1-x_2\sqrt{a} \end{pmatrix}, \tag{3}$$

where x_1,x_2,x_3,x_4,a,b belong to the above-mentioned field and a,b are chosen such that the determinant of this matrix is nonzero for any $x_i\neq 0$ from the field.

If this determinant is indefinite (i.e., can be both positive and negative) but under any of $n-1$ nontrivial automorphisms becomes of definite sign, then the set of matrices (3) with unit determinant where x_1,x_2,x_3,x_4 are, roughly speaking, *integers* of the field defines a discrete group of fractional transformations of the upper half plane [17,18].

All such groups and also all groups having a subgroup (of finite index) in common with one of those groups are called arithmetic groups. Knowing the discrete group one can find its fundamental domain [14,19]. In general it will be a polygon (each side of it being a geodesic). In Ref. [20] all triangles (m,n,p) tessellating the plane by reflections under the action of an arithmetic group were explicitly enumerated. This list contains 85 triangles, and it includes *all three tiling triangles considered so far* whose quantum spectra possess nongeneric behavior.

The very important property of arithmetic groups is that the arithmetic structure inherent in their construction manifests itself in an unexpectedly large multiplicity of lengths of periodic orbits. To see it we shall need the general criterion of arithmeticity proved in [21].

Let γ be any matrix of a group. This group will be an arithmetic group if and only if the following conditions are satisfied: (i) All $\mathrm{tr}(\gamma)$ are contained in the field of integers of a certain real algebraic field of finite degree. (ii) For any automorphism of this field which changes $|\mathrm{tr}\gamma|$ the value of the transformed trace satisfies the inequality $|\mathrm{tr}\gamma|\leq 2$.

The meaning of this criterion is the following. For an arithmetic group, $t_0=\mathrm{tr}(\gamma)$ should have the form (2) where all m_i are, roughly speaking, integers and u is a *fixed* root of Eq. (1). For fields of degree n one has $n-1$ nontrivial automorphisms $u \rightarrow u_j$ and the trace of the matrix becomes

$$t_j = m_0 + m_1 u_j + \cdots + m_{n-1} u_j^{n-1} . \quad (4)$$

According to the above criterion only two possibilities are allowed for arithmetic groups: all $|t_j| = t_0$ or all $|t_j| \le 2$.

Considering t_0 and t_j as given, one can solve the above equations for the m_i: $m_i = A_{i0} t_0 + A_{ij} t_j$. (The determinant is nonzero as all u's are different.)

Taking into account that values of m_i should be integers (or proportional to integers) one concludes that $t_0 = \mathrm{tr}(\gamma)$ takes only a specific set of values:

$$t_0 = a_i n + \beta_i(\gamma) , \quad (5)$$

where n is an integer, a_i are constants, and $|\beta_i(\gamma)| <$ const for all γ. Besides, there is only a finite set of such relations.

The geometrical length of a periodic orbit (l) is connected with the trace of the group matrix by well-known relations [7]: $\exp(l/2) = |\mathrm{tr}\gamma| + O(\exp(-l/2))$. From Eq. (5) one concludes that as $L \rightarrow \infty$, the number of different lengths of periodic orbits with $l < L$ is proportional to the number of integers less than $\exp(L/2)$:

$$N_{\text{diff. lengths}}(l < L) \sim \text{const} \times \exp(L/2) . \quad (6)$$

It is known [22] that for any hyperbolic model the total number of periodic orbits with lengths less than a given value is $N_{\text{tot}}(l < L) \sim \exp(L)/L$ as $L \rightarrow \infty$.

Let $g(l)$ be the multiplicity of periodic orbits with length l; then these equations imply that the mean multiplicity is

$$\langle g \rangle = \frac{\sum_{l<L} g(l)}{\sum_{l<L} 1} \sim \text{const} \times \frac{e^{L/2}}{L} . \quad (7)$$

We see that the arithmetic nature of the groups leads to exponentially large multiplicities of the lengths of periodic orbits. For the triangle (2,3,8) this property was observed numerically in [23] and has been proved analytically in [10,24].

In general, one does not expect such a large exact degeneracy of periodic orbits, but one should keep in mind that for any Riemann surface, multiplicities of periodic orbit lengths are always unbounded [25], though not so large as in Eq. (7).

In [26] it was shown that the form factor of the two-point correlation function of energy levels in diagonal approximation can be expressed as a sum over all periodic orbits:

$$K^{\text{diag}}(t) = \sum_{T_p} |A_p|^2 g(T_p)^2 \delta(t - T_p) , \quad (8)$$

where $g(T_p)$ is the degeneracy of periodic orbits with period T_p. In our cases, as $l_p \rightarrow \infty$, $A_p = l_p \exp(-l_p/2)/k\pi$ [7], and $T_p = l_p/k$, where l_p is the geometrical length of the trajectory and k is the momentum. The validity of the diagonal approximation is restricted to small values of t: $1/k \ll t \le t_E$, when one can ignore the interference between trajectories with *different* actions. The time t_E is called the Ehrenfest time and in our case it can be estimated as follows. To be sure that after a smoothing which is implicitly assumed in Eq. (8) the contribution from off-diagonal terms is small, the difference in actions between nearby trajectories should be not too small: $\Delta S/\hbar \ge$ const. Using (6) one concludes that it is necessary to include all trajectories up to $l_{\max}$ determined from the equation $k \exp(-l_{\max}) =$ const which is equivalent to $t_E = l_{\max}/k \sim (\ln k)/k$.

For generic systems the multiplicity comes exclusively from the time-reversal orbits and one obtains [26] a constant slope for $K(t)$ which is in agreement with standard RME. But for arithmetic systems the exponentially large exact multiplicities given by Eq. (7) lead to

$$K^{\text{diag}}(t) = \text{const} \times e^{kt/2}/k . \quad (9)$$

So the form factor $K(t)$ for arithmetic systems grows much faster than was usually assumed and within a time t^* of order of the Ehrenfest time it becomes of the order of 1. It is known [26] that for $t \rightarrow \infty$, $K(t) \rightarrow \bar{d}$, where $\bar{d}$ is the mean level density (which is a finite constant for hyperbolic models with discrete spectra).

As $k \rightarrow \infty$ a good approximation for $K(t)$ is just the step function:

$$K(t) = \bar{d}\Theta(t - t^*) . \quad (10)$$

For Poisson statistics $K(t)$ should always equal $\bar{d}$; therefore, for arithmetic systems the two-point form factor, within applicability of the diagonal approximation, quickly jumps just to the universal saturation value. Though we cannot exclude oscillations near this value, the form factor for arithmetic systems is undoubtedly much closer to the Poisson prediction typical for integrable systems than to any of standard RME conjectured for generic ergodic systems.

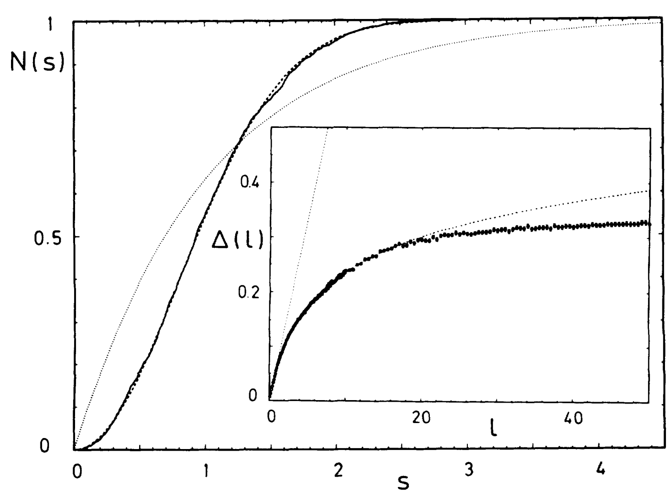

FIG. 2. The same as Fig. 1 but for the triangle $(2,5,\infty)$.

It has been shown above that for arithmetic systems, due to the existence of many periodic orbits with exactly the same lengths, the statistical properties of the spectrum deviate from predictions of the random-matrix theory and are close to the ones typical of integrable systems. Since the number of arithmetic triangles is finite, most of the tessellating triangles are nonarithmetic. We consider one such triangle, namely, the triangle with angles $(\pi/2, \pi/5, 0)$, and compute 1700 levels with Dirichlet boundary conditions. The results are presented in Fig. 2. It is seen that the fluctuation properties of this triangular billiard are in a good agreement with GOE predictions (at least, much better than for an arithmetic system as in Fig. 1).

Yet the numerically computed degeneracy of lengths of periodic orbits for this triangle seems to be growing, but its increasing rate is much smaller than for arithmetic systems. Whether it means that level statistics at higher energy will be nonuniversal even in this case requires further investigation.

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