

**Problem of Light Diffusion in Strongly Scattering Media**

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Recently, in connection with the problem of Anderson localization of light, a new physical mechanism was considered for the decrease of the diffusion coefficient, in which it is supposed that the velocity of energy transport, which appears in the diffusion constant  $D = v_E l_{tr}/3$ , may be much smaller than the phase velocity. Using an exact definition of the diffusion coefficient and generalized Ward identity, we, however, show that in the low-density approximation  $v_E$  coincides with the phase velocity.

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There is a considerable interest now in the propagation and multiple scattering of light in strongly disordered media with parameters which approach the assumed threshold of Anderson localization of light, where  $\lambda/l \leq 1$  and the diffusion coefficient  $D$  must tend to zero [1]. In a recent Letter, van Albada, van Tiggelen, Lagendijk, and Tip [2] presented the results of measurements of the diffusion coefficient and the transport mean free path  $l_{tr}$  which demonstrate that in a strongly scattering random

dielectric medium the velocity of electromagnetic energy transport  $v_E$ , defined as

$$D = v_E l_{tr}/3, \tag{1}$$

can be very much lower than the phase velocity  $c_{ph}$ . To explain this discrepancy the authors developed a theory based on the low-density approximation of the Bethe-Salpeter equation. They used an approach similar to that employed by Vollhardt and Wölfle [3] to derive the following expression:

$$v_E = c^2/(1 + \alpha)c_{ph}, \quad \alpha = n \left\langle I(\mathbf{s}) \frac{d\phi(\mathbf{s})}{dp} \right\rangle_s - n \frac{d}{dp^2} \text{Re}l(\mathbf{p}, \mathbf{p}; p^2), \tag{2}$$

where  $t(\mathbf{p}, \mathbf{p}'; E)$  is the element of the  $T$  matrix for one scatterer,  $I(\mathbf{s})$  and  $\phi(\mathbf{s})$  denote the scattering intensity and the phase shift in the direction of the unit vector  $\mathbf{s}$ , and  $\langle \dots \rangle_s$  corresponds to angular averaging.

In this Letter we examine these formulas on the basis of the exact Green-Kubo expression for the diffusion coefficient of wave propagation in nonabsorbing random media and the Ward identity generalized to the case of the scalar wave equation or Schrödinger equation with an

energy-dependent potential [4]. We also treat this expression in the framework of the low-density approximation and arrive at the conclusion that the correct expression for  $v_E$  coincides with the phase velocity.

According to Ref. [4] the diffusion constant can be written as

$$D = (N_0/N)D_K, \tag{3}$$

where  $D_K$  is represented in the usual Kubo form, viz.,

$$D_K = \frac{c^2}{3\pi EN_0} \int_{\mathbf{p}} \left[ \int_{\mathbf{p}'} (\mathbf{p} \cdot \mathbf{p}') \Phi_{\mathbf{p}\mathbf{p}'}(0,0) - p^2 \frac{\partial}{\partial p^2} \text{Re}G_{\mathbf{p}} \right]. \tag{4}$$

Here we denote  $\int_{\mathbf{p}} \equiv (2\pi)^{-3} \int d\mathbf{p}$ ,  $c$  is the phase velocity in a homogeneous medium,  $G_{\mathbf{p}} \equiv G_{\mathbf{p}}(E - i0) = [k_0^2 - p^2 - M_{\mathbf{p}}(E - i0)]^{-1}$  with  $k_0 = E/c$  the Fourier transform of the averaged Green's function, and  $\Phi_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega)$  satisfies the Bethe-Salpeter equation given by

$$[-iE\omega/c^2 + i(\mathbf{p} \cdot \mathbf{q})] \Phi_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega) = \Delta G_{\mathbf{p}}(\mathbf{q}, \omega) \delta_{\mathbf{p}\mathbf{p}'} + \int_{\mathbf{p}''} U_{\mathbf{p}\mathbf{p}''}(\mathbf{q}, \omega) \Phi_{\mathbf{p}''\mathbf{p}'}(\mathbf{q}, \omega), \tag{5}$$

where  $\delta_{\mathbf{p}\mathbf{p}'} \equiv (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}')$ .  $2i\Delta M_{\mathbf{p}}(\mathbf{q}, \omega) = M_{\mathbf{p}_-}(E_-) - M_{\mathbf{p}_+}(E_+)$  with  $\mathbf{p}_{\pm} = \mathbf{p} \pm \mathbf{q}/2$  and  $E_{\pm} = E \pm (\omega/2 + i0)$ ;  $\Delta G_{\mathbf{p}}(\mathbf{q}, \omega)$  is constructed in the same way. The function  $U_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega)$  is defined by

$$U_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega) = \Delta G_{\mathbf{p}}(\mathbf{q}, \omega) K_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega) - \Delta M_{\mathbf{p}}(\mathbf{q}, \omega) \delta_{\mathbf{p}\mathbf{p}'}. \tag{6}$$

With  $N_0 = \pi^{-1} \int_{\mathbf{p}} \text{Im}G_{\mathbf{p}}(E)$ , the density of states for the considered problem is  $N = N_0(1 + a)$  where

$$N_0 a = -\pi^{-1} \int_{\mathbf{p}} A_{\mathbf{p}}(0,0) \text{Im}G_{\mathbf{p}}. \tag{7}$$

The function  $A_p(\mathbf{q}, \omega)$  takes the form

$$i\omega EA_p(\mathbf{q}, \omega)/c^2 = \int_{\mathbf{p}'} U_{\mathbf{p}'\mathbf{p}}(\mathbf{q}, \omega). \quad (8)$$

[Note that in the text of Ref. [4] a correction is needed: There should be a change in sign in the definition of the constant  $a$ , Eq. (17).]

The key point of our derivation is the generalized Ward identity [4], written as

$$g(E-)M_{\mathbf{p}_+}(E+) - g(E+)M_{\mathbf{p}_-}(E-) = \int_{\mathbf{p}'} K_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega) [g(E+)G_{\mathbf{p}'}(E+) - g(E-)G_{\mathbf{p}'}(E-)], \quad (9)$$

where  $g(E) = E^2$  [and  $g(E) = \text{const}$  for the problem of electron-impurity interaction]. Since Eq. (9) for  $q=0$ ,  $\omega=0$  yields the usual Ward identity, the expansion at  $q=0$  to first order in  $\omega$  gives

$$\frac{\partial}{\partial E} \text{Re}M_{\mathbf{p}} + \int_{\mathbf{p}'} \left[ 2i \left[ \frac{\partial}{\partial \omega} K_{\mathbf{p}\mathbf{p}'}(0, \omega) \right]_{\omega=0} \text{Im}G_{\mathbf{p}'} - K_{\mathbf{p}\mathbf{p}'}(0, 0) \frac{\partial}{\partial E} \text{Re}G_{\mathbf{p}'} \right] = \frac{2}{E} \left[ \text{Re}M_{\mathbf{p}} + \int_{\mathbf{p}'} K_{\mathbf{p}\mathbf{p}'}(0, 0) \text{Re}G_{\mathbf{p}'} \right]. \quad (10)$$

It is easily seen that such an expansion of the right-hand side of Eq. (8) leads to the equality

$$A_p(0, 0) = k_0^{-2} \left[ \text{Re}M_{\mathbf{p}} + \int_{\mathbf{p}'} K_{\mathbf{p}\mathbf{p}'}(0, 0) \text{Re}G_{\mathbf{p}'} \right]. \quad (11)$$

Hence, by using the Ward identity we find, from Eq. (7), that

$$1 + a = \frac{1}{\pi N_0} \int_{\mathbf{p}} \left[ \frac{p^2}{k_0^2} \right] \text{Im}G_{\mathbf{p}}. \quad (12)$$

Now we consider the low-density approximation [2,5] in which the imaginary part of the Green's function has a sharp peak at  $p = E/c_{\text{ph}}$ ,

$$\text{Im}G_{\mathbf{p}} \cong \pi \delta[p^2 - (E/c_{\text{ph}})^2]. \quad (13)$$

where  $E/c_{\text{ph}} \cong k_0 - \text{Re}M_{k_0}/2k_0$ . To calculate the coefficient  $D_K$ , we solve, using Eq. (13), the equation for the vector  $\Phi_{\mathbf{p}}^{(i)} = \int_{\mathbf{p}'} p_i' \Phi_{\mathbf{p}\mathbf{p}'}(0, \omega)$ ,  $\omega \rightarrow 0$ , following from (5), and then substitute the solution into Eq. (4) to obtain

$$D_K = \frac{1}{3} c_{\text{ph}} l_{\text{tr}} (c/c_{\text{ph}})^2, \quad (14)$$

where  $l_{\text{tr}} = l/(1 - \langle \cos \vartheta \rangle)$  and  $1/l = \text{Im}M_{k_0}(E)/k_0$ . Here we omit the contribution of the second term on the right-hand side of Eq. (4) which is negligible in the low-density limit.

The comparison of Eqs. (1), (3), and (14) yields

$$\frac{v_E}{c_{\text{ph}}} = \frac{(c/c_{\text{ph}})^2}{1+a}. \quad (15)$$

On the other hand, from Eqs. (12) and (13) it follows

$$M_{\mathbf{p}}(E-i0) \cong nt(\mathbf{p}, \mathbf{p}; E-i0),$$

$$K_{\mathbf{p}\mathbf{p}'}(0, \omega) \cong nt(\mathbf{p}, \mathbf{p}'; E-\omega/2-i0)t(\mathbf{p}, \mathbf{p}'; E+\omega/2+i0),$$

where  $n$  is the number density of scatterers. By representing the scattering amplitude in the form  $t(\mathbf{p}, \mathbf{p}'; E \pm i0) = |t| \exp(\pm i\phi)$ , one can obtain, with the aid of Eq. (13),

$$a = -\frac{cn}{2k_0} \frac{\partial}{\partial E} \text{Re}t(k_0\mathbf{s}, k_0\mathbf{s}; E \pm i0) + \frac{cn}{4\pi} \left\langle |t(k_0\mathbf{s}, k_0\mathbf{s}'; E \pm i0)|^2 \frac{\partial}{\partial E} \phi(k_0\mathbf{s}, k_0\mathbf{s}'; E) \right\rangle_s + a', \quad (18)$$

where

$$a' = \frac{cn}{2\pi k_0 N_0} \int_{\mathbf{p}} \text{Im}t(\mathbf{p}, \mathbf{p}; E-i0) \frac{\partial}{\partial E} \text{Re}G_{\mathbf{p}}^{(0)}, \quad (19)$$

that  $1+a = (c/c_{\text{ph}})^2$ . Thus, we see that in the low-density limit,  $v_E$  coincides with  $c_{\text{ph}}$ .

Now we would like to analyze briefly the hydrodynamic approach [3] used in Ref. [2]. Let us introduce the energy density  $\Phi(\mathbf{q}, \omega)$  and flux  $\mathbf{J}(\mathbf{q}, \omega)$  as the integrals over  $\mathbf{p}, \mathbf{p}'$  of the function  $\Phi_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega)$  multiplied by 1 and  $\mathbf{p}$ , respectively. Then, integration of Eq. (5) leads to the continuity equation with a supplementary term that is the integral of the second term on the right-hand side of Eq. (5). For  $q, \omega \rightarrow 0$  we take into account that the low-density approximation for  $\Phi_{\mathbf{p}\mathbf{p}'}(\mathbf{q}, \omega)$  is proportional to  $\text{Im}G_{\mathbf{p}}$ , given by Eq. (13), and use Eqs. (7), (8) to find

$$-i(E\omega/c^2)(1+a)\Phi(\mathbf{q}, \omega) + i\mathbf{q} \cdot \mathbf{J}(\mathbf{q}, \omega) = \pi N_0. \quad (16)$$

Further, following Ref. [3] we treat the equation for the flux, derived from Eq. (5), and obtain Fick's law  $\mathbf{J}(\mathbf{q}, \omega) \cong -i\mathbf{q} D_K (E/c^2) \Phi(\mathbf{q}, \omega)$ , with  $D_K$  defined by Eq. (14). Hence, the diffusion constant takes the form given by Eq. (3) and we arrive again at the formula (15).

We have evaluated the low-density limit for the constant  $a$  in a simple way, and relied on Eqs. (11), (12). However, the calculations can be performed using Eq. (7) and the expression for the function  $A_p(0, 0)$  which follows from Eq. (11) and the left-hand side of Eq. (10). In the low-density approximation [2,5] the mass operator and irreducible vertex take the form

(17)

with  $G_{\mathbf{p}}^{(0)} = [(E - i0)^2/c^2 - p^2]^{-1}$ . Since  $E = ck_0$  it is immediately seen that the first two terms on the right-hand side of Eq. (18) coincide with Eq. (2). It is also clear that the term  $a'$ , which corresponds to the third term on the left-hand side of Eq. (10), is lost in the definition of  $a$ . Further, it is worth pointing out that all terms in Eq. (18) are formally of the same order in density. And finally, there exists a generalized unitary relation for the  $T$  matrix, which can be derived, for instance, by substituting Eqs. (17) into (9) and changing  $G_{\mathbf{p}}(E)$  to  $G_{\mathbf{p}}^{(0)}(E)$ . From this relation it follows that in the case of electron-impurity interaction [ $g(E) = \text{const}$ ], the right-hand side of Eq. (18) (but not the sum of the first and second terms, as it is affirmed in Ref. [2]) is equal to zero. On the other hand, using this relation for the considered case of wave scattering, we can represent Eq. (18) in the form of the integral of  $\text{Im}(t(\mathbf{p}, \mathbf{p}; E)G_{\mathbf{p}}^{(0)})$  over  $\mathbf{p}$ . This expression, when treated within the same approximation that leads to Eq. (13), results, as expected, in the

equality  $a = (c/c_{\text{ph}})^2 - 1$ , obtained above.

Thus, it is shown that the correct low-density limit for  $v_E$  coincides with the phase velocity. We conclude that the formula (2) is not a result of the successive low-density (or pure Boltzmann, in the terminology of Ref. [2]) approximation. Therefore, the agreement of Eq. (2) with the experimental measurements, presented in Ref. [2], can be regarded as a successful heuristic result only.

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