QCD Perturbation Theory at Low Energies

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We apply the optimization procedure based on the principle of minimal sensitivity to the third-order calculation of $R_{e^+e^-}$. The effective couplant remains finite, freezing to a value $\alpha_s/\pi=0.26$ at low energies. Using Poggio-Quinn-Weinberg smearing we find good agreement between theory and experiment right down to zero energy.

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Does QCD perturbation theory have anything at all to say about physics below 1 GeV? Since the effective couplant

$$a\equiv\frac{\alpha_s(\mu)}{\pi}\approx\frac{1}{b\ln(\mu/\Lambda)} \tag{1}$$

[with $b=(33-2N_f)/6$] becomes infinite when $\mu=\Lambda\approx$ (a few hundred MeV), it is generally thought that perturbation theory must completely break down in this region. However, Eq. (1) is valid only to leading order: If higher orders produce a nontrivial zero of the β function at $a=a^*$, then a would remain finite, "freezing" to the fixed-point value a^* as $\mu\to 0$.

Is this "fixed-point scenario" realized? Seemingly not, since the calculated coefficients of the β function [1],

$$\mu\frac{\partial a}{\partial\mu}\equiv\beta(a)=-ba^2(1+ca+c_2a^2+\cdots), \tag{2}$$

$$c=\frac{153-19N_f}{2(33-2N_f)}, \tag{3}$$

$$c_2(\overline{\mathrm{MS}})=\frac{3}{16}\frac{1}{33-2N_f}\left[\frac{2857}{2}-\frac{5033}{18}N_f+\frac{325}{54}N_f^2\right], \tag{4}$$

are both positive for fewer than six flavors. However, while b and c are invariants, c_2 depends upon the renormalization scheme (RS). The c_2 quoted above is for the "modified minimal subtraction" ($\overline{\mathrm{MS}}$) scheme. In other RS's, though, c_2 may be large and negative—indeed, one is free to *define* the RS such that this is true—so it is clear that the fixed-point issue cannot be decided from β alone.

Instead, one should examine the low-energy behavior of some physical quantity, such as the QCD correction to the e^+e^- total cross section at a c.m. energy Q:

$$\mathcal{R}(Q)=a(1+r_1a+r_2a^2+\cdots). \tag{5}$$

The issue must also be addressed in the context of a proper resolution of the RS dependence problem—which, in our view, means within the framework of "optimized perturbation theory" (OPT) [2]. This is founded upon the observation that, while perturbative *approximations* to physical quantities do depend upon RS, the exact result is RS invariant, so the approximation is only believable where it is stable to small changes in RS. This "principle of minimal sensitivity" is supported by many examples [2] and it has an excellent track record in QCD phenomenology [3].

An OPT analysis of the fixed-point issue was given in Ref. [4]. It was found that a complete third-order calculation is a prerequisite for addressing this question (and at that time no such calculation existed in the QCD case). To third order the fixed point is determined by the equation

$$\tfrac{7}{4}+ca^*+3\rho_2a^{*2}=0, \tag{6}$$

where ρ_2 is the RS-invariant combination

$$\rho_2\equiv r_2+c_2-(r_1+\tfrac{1}{2}c)^2. \tag{7}$$

Thus, for $c>0$, the existence or nonexistence of a fixed point (in third order) is governed by the sign of ρ_2. If ρ_2 is positive then there is no solution for a^*, meaning that third-order perturbation theory breaks down before one reaches zero energy, just as in lower orders. However, if ρ_2 is negative then a positive root a^* exists, meaning that OPT yields a finite result down to zero energy. If the a^* is sufficiently small, then—with appropriate caveats about nonperturbative effects—one can take the perturbative result seriously.

In 1988 a calculation of the third-order coefficient r_2 was reported [5]. Later, it was found that this result was in error, and a corrected result was published in 1991 [6]. The old result gave a positive ρ_2, indicating no hope for perturbation theory at low energies. Much worse, though, the ρ_2 was so large (65 for four flavors) that even at relatively high energies the third-order corrections were disconcertingly large. Several authors noted this as evidence against the usefulness of "optimization" [7]. However, with the new result for r_2, the situation is transformed: The new ρ_2 is negative, and of moderate size (-13.9 for four flavors) [8]. At high energies the net third-order correction is quite small [9], and one finds fixed-point behavior at low energies [10,11].

In this Letter we report on the results of our investigation of third-order OPT applied to $\mathcal{R}$ in e^+e^- and its comparison, using the Poggio-Quinn-Weinberg (PQW) [12] smearing method, to experimental data in the region $0<Q<6$ GeV. Fuller details will be given in a separate

paper [13]. We use standard values for the current quark masses ($m_u = 5.6$ MeV, $m_d = 9.9$ MeV, $m_s = 199$ MeV, $m_c = 1.35$ GeV) and a Λ parameter that corresponds to $\Lambda_{\overline{\rm MS}} = 230$ MeV for four flavors [14].

The matching of Λ across flavor thresholds requires comment. The main effect comes from requiring consistency in the definition of Λ so that a is continuous at a threshold. The point is well explained by Marciano [15], but unfortunately his analysis uses a truncated expansion of a in powers of $1/\ln(\mu/\Lambda)$. Avoiding this unnecessary approximation (which would ruin any attempt to go to low energies), we proceed as follows: The integrated β-function equation can be written as [2]

$$\ln(\mu/\tilde{\Lambda}) = \int_0^{a(\mu)} \frac{dx}{\beta(x)} - \int_0^{\infty} \frac{dx}{\beta^{(2)}(x)} \equiv \frac{\hat{K}(a)}{b}, \quad (8)$$

where $\beta^{(2)}(x)$ is the second-order truncation, $-bx^2(1+cx)$, of the β function. [Our $\tilde{\Lambda}$ is related [2] to the conventional Λ parameter by $\ln(\Lambda/\tilde{\Lambda}) = (c/b)\ln(2c/b)$.] Consider a theory with $n+1$ quark flavors, but where we are below the threshold of one quark. We could either use Eq. (8) with $N_f = n$, thus defining a $\tilde{\Lambda}$ for n flavors, or use (8) with $n+1$ flavors, breaking the first integration into two parts:

$$\ln(\mu/\tilde{\Lambda}_+) = \int_0^{a_{\rm th}} \frac{dx}{\beta_+(x)} + \int_{a_{\rm th}}^{a(\mu)} \frac{dx}{\beta_-(x)} - \int_0^{\infty} \frac{dx}{\beta_{\mp}^{(2)}(x)}, \quad (9)$$

where the + and − subscripts mean "above" and "below" threshold (i.e., $N_f = n+1$ and $N_f = n$), respectively. Requiring this to agree with Eq. (8) with $N_f = n$, we obtain the matching condition

$$\ln(\tilde{\Lambda}_+/\tilde{\Lambda}_-) = \frac{\hat{K}_-(a_{\rm th})}{b_-} - \frac{\hat{K}_+(a_{\rm th})}{b_+}. \quad (10)$$

The energy at which N_f should be incremented is not unambiguously defined. We elected to make the changeover at the $q\bar{q}$ threshold, so that $a_{\rm th}$ is the optimized couplant obtained at $Q = 2m_q$ [16].

The optimization procedure in third order is described in Ref. [2], and details of its implementation are given in Refs. [10,13]. Briefly, the third-order approximant $\mathcal{R}^{(3)}$ [defined by truncating (2) and (5) after three terms] depends on RS through two parameters, $\tau \equiv b\ln(\mu/\tilde{\Lambda})$ and c_2. The coefficients r_1 and r_2 depend on these RS parameters such that the combinations $\rho_1 \equiv \tau - r_1$ and ρ_2 [Eq. (7)] are invariant. The values of these invariants can be computed from the results obtained in the $\overline{\rm MS}(\mu = Q)$ scheme [6]:

$$r_1(\overline{\rm MS};\mu = Q) = 1.9857 - 0.1153N_f, \quad (11)$$

$$r_2(\overline{\rm MS};\mu = Q) = -6.6368 - 1.2001N_f - 0.0052N_f^2 - 1.2395\left(\sum q_i\right)^2 \Big/ \left(3\sum q_i^2\right), \quad (12)$$

with $\tau(\overline{\rm MS};\mu = Q) = b\ln(Q/\tilde{\Lambda}_{\overline{\rm MS}})$, and $c_2(\overline{\rm MS})$ as given above. (Note that, for a fixed N_f, ρ_2 is a fixed number, while ρ_1 is a function of the c.m. energy Q.) The principle of minimal sensitivity picks out an "optimum" scheme in which the RS-stability conditions $\partial\mathcal{R}^{(3)}/\partial\tau = 0$ and $\partial\mathcal{R}^{(3)}/\partial c_2 = 0$ are satisfied. These equations together with the definitions of ρ_1 and ρ_2 and the integrated β-function equation (8) allow one to solve for the optimized couplant $\bar{a}$ and the optimized coefficients $\bar{c}_2, \bar{r}_1, \bar{r}_2$. The procedure requires the numerical solution of two coupled equations. A good initial guess at the solution can be obtained from the approximation [17] $\bar{r}_2 = -\frac{1}{3}\bar{c}_2$, $\bar{r}_1 = 0$ (or, better, $\bar{r}_1 = \frac{1}{3}\bar{c}_2\bar{a}$). At energies below $\tilde{\Lambda}$ this approximation becomes poor, and the numerical solution of the equations becomes quite delicate (see [10,13]). However, in this region one can be guided by the fixed-point solution [4] $\bar{r}_2 = -\frac{2}{3}\bar{c}_2$, $\bar{r}_1 = \frac{1}{2}\bar{c}_2 a^*$.

For the effective couplant and for $\mathcal{R}^{(3)}$ itself we obtain the results shown in Fig. 1. Below 300 MeV the effective α_s/π is essentially constant at the value 0.263, which is the fixed-point value obtained from (6) with two massless flavors. Note that this number is independent of the Λ value. This α_s/π value is determined purely by optimized perturbative QCD with no experimental input. It agrees remarkably well with low-energy α_s/π values determined phenomenologically [18,19].

To obtain $R_{e^+e^-}$, allowing for quark masses, we use the formula [12]

$$R_{e^+e^-} = 3\sum_i q_i^2 T(v_i)[1 + g(v_i)\mathcal{R}], \quad (13)$$

where

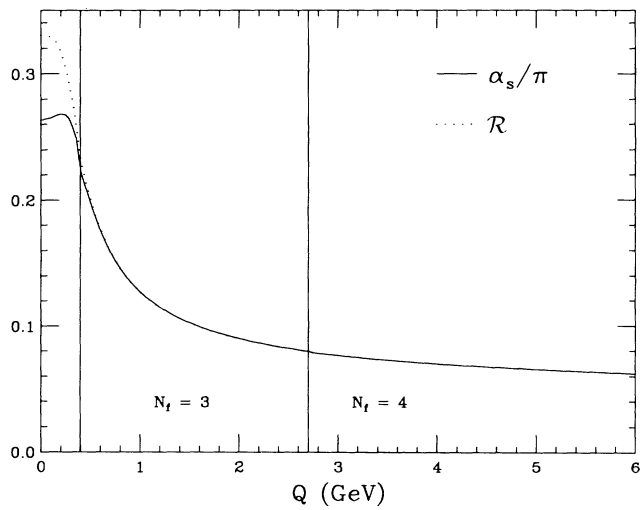

FIG. 1. The optimized couplant $\bar{a}$ $(= \alpha_s/\pi)$ obtained from third-order optimization of $\mathcal{R}$, where $R_{e^+e^-} = 3\sum q_i^2(1+\mathcal{R})$ for massless quarks, and $R_{e^+e^-}$ is the e^+e^- total hadronic cross section normalized by the cross section to $\mu^+\mu^-$. The vertical lines indicate the s and c quark thresholds.

$$v_i=(1-4m_i^2/Q^2)^{1/2},$$
$$T(v)=v(3-v^2)/2, \tag{14}$$
$$g(v)=\frac{4\pi}{3}\left[\frac{\pi}{2v}-\frac{3+v}{4}\left(\frac{\pi}{2}-\frac{3}{4\pi}\right)\right].$$

Except for the use of an effective N_f in obtaining $\mathcal{R}$, we are ignoring mass dependence in the coefficients r_1, r_2 because the calculations have been done only for massless quarks.

A direct comparison of the resulting $R_{e^+e^-}$ with experiment is not possible because nonperturbative effects drastically change the threshold behavior. However, PQW [12] argue that one can define a suitably smeared quantity,

$$\bar{R}_{\rm PQW}(Q;\Delta)\equiv\frac{\Delta}{\pi}\int_0^\infty ds'\frac{R_{e^+e^-}(\sqrt{s'})}{(s'-Q^2)^2+\Delta^2}, \tag{15}$$

which is insensitive to nonperturbative effects provided that Δ is large enough to smooth out the threshold resonances. PQW used $\Delta=3$ GeV^2. Applying this smearing both to the perturbative prediction and to the experimental data [20], we obtain Fig. 2. There is some discrepancy in the charm-threshold region, but below 2 GeV the theoretical and experimental curves are almost indistinguishable. If we reduce the amount of smearing, taking $\Delta=1$ GeV^2, we obtain Fig. 3. Here the smearing is not enough to smooth out the structure in the charm-threshold region, but is sufficient to smooth out the ρ, ω, and ϕ resonances, and the agreement below 1 GeV remains very good.

There are uncertainties in the data and in the theoretical prediction, of course. On the experimental side there are (10–20)% normalization uncertainties in the continuum multihadron data and about 15% uncertainties in the J/ψ and ψ' resonance parameters [20]. This (and uncertainty in the c quark mass) can account for much of the discrepancy in the charm region. However, $R_{e^+e^-}$ below 1.2 GeV is dominated by ρ, ω, and ϕ resonances, whose parameters are known to 5% or better [14,20]. We estimate that the experimental $\bar{R}_{\rm PQW}$ curve below 1 GeV is trustworthy to roughly ± 0.12 for $\Delta=3$ GeV^2, and to about ± 0.07 for $\Delta=1$ GeV^2.

On the theory side there is, of course, an uncertainty due to the truncation of the perturbation series. Above 1 GeV one can see that this error is small by comparing second- and third-order optimized results. Below about 1 GeV it is undeniable that the prediction for the net QCD correction term $\mathcal{R}$ becomes rather uncertain. For instance, at $Q=0$ the optimized coefficients become $\bar{r}_1=-2.9$, $\bar{r}_2=14.6$, with $\bar{a}=0.26$, giving a series $\mathcal{R}(0)=0.26(1-0.76+1.01)$ in which the higher-order terms are comparable to the leading term. However, at least the signs alternate, and the corrections are not huge. We would say that the resulting $\mathcal{R}$ prediction may well be off by a factor of 2, but is unlikely to be off by an order of magnitude. Thus, although its precise value is uncertain, $\mathcal{R}$ near $Q=0$ is rather small (0.3 ± 0.3, say), so that the uncertainty in $R_{e^+e^-}$ itself is modest, and smearing further dilutes the uncertainty. There are other uncertainties, particularly from the quark masses. Altogether we estimate that the theoretical $\bar{R}_{\rm PQW}$ curve below 1 GeV is trustworthy to roughly ± 0.07 for either Δ value.

How significant is the good agreement between the data and the OPT prediction? As we just argued, the smeared result is rather insensitive to $\mathcal{R}$ because it is small. However, the important point is that OPT *predicts* that $\mathcal{R}$ is small *down to* $Q=0$. One could well have imagined that the QCD correction term became large at low energies. As an illustration, consider a "straw-man" theory in which $\mathcal{R}$ is the same as ours down to 2 GeV, but then continues to rise as $1/\{\frac{9}{2}\ln(Q/\Lambda_0)\}$, with

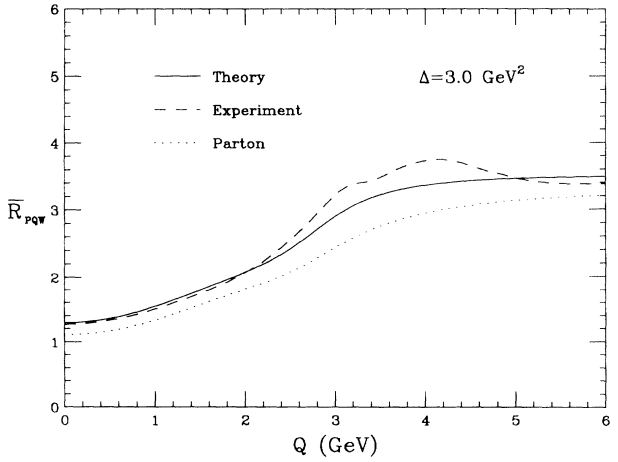

FIG. 2. The PQW-smeared R ratio for $\Delta=3$ GeV^2, showing the prediction of "optimized" QCD perturbation theory compared to experiment. Also shown is the prediction of the naive parton model [i.e., $\mathcal{R}=0$ in Eq. (13)].

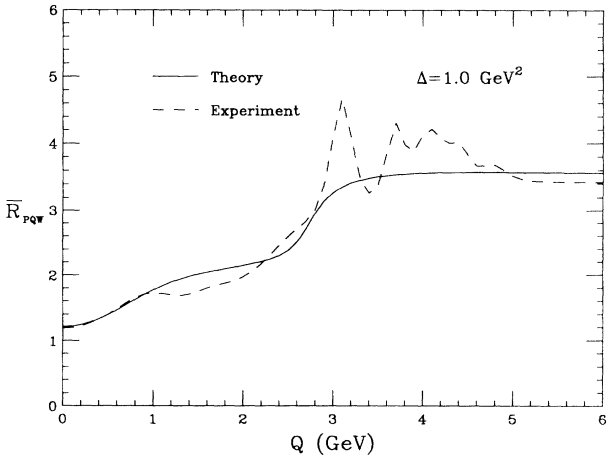

FIG. 3. The PQW-smeared R ratio for $\Delta=1$ GeV^2.

$\Lambda_0 \approx 0.2$ GeV, until it reaches some value H, at which value it remains frozen down to $Q=0$. Such a theory gives essentially the same $\bar{R}_{\mathrm{PQW}}$ prediction as ours if H is small (≈ 0.3), but for larger H it gives a result that is too large at low Q. Based on our previous estimate of the experimental uncertainty in $\bar{R}_{\mathrm{PQW}}(\Delta=1\ \mathrm{GeV}^2)$ we can say that the data imply a limit $H \leq 2$. Thus, the data can rule out a large $\mathcal{R}$ term.

One could also ask: How predictive is the theory? How different would the low-energy data have to be to give a significant disagreement with the predicted $\bar{R}_{\mathrm{PQW}}$ near $Q=0$? The data below 1 GeV are dominated by the ρ peak, which, after smearing with $\Delta=1\ \mathrm{GeV}^2$, contributes a roughly constant 0.7–0.8 to $\bar{R}_{\mathrm{PQW}}$ below 1 GeV. Thus, a 10% change in the area under the ρ peak would change $\bar{R}_{\mathrm{PQW}}$ by the ± 0.07 estimated uncertainty in the theoretical prediction. We conclude that perturbative QCD can tell us, at least crudely, the size of the ρ resonance.

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