Disassembling Anyons

Frank Wilczek
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, New Jersey 08540
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A notion of mutual fractional statistics is introduced, and a field-theoretic construction to implement it is given. Realizations in layered Hall systems are suggested. A generalization of anyon superconductivity, arising from mutual statistics, is proposed.

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As is by now well known, one may transmute the statistics of particles in 2+1 dimensions continuously by afixing fictitious charge and localized flux to them [1]. What could be more natural than to break this procedure up into two more elementary operations—attaching fictitious charge to one kind of particle, and fictitious flux to another? That is what is considered in this Letter.

(1) Chern-Simons construction.—Suppose that we have a (2+1)-dimensional system with two conserved species (say, for example, two gases of free point particles). Let us consider adding to the Lagrangian for this system the terms

$$\Delta\mathcal{L}=qj^{(1)}\cdot a^{(1)}+qj^{(2)}\cdot a^{(2)}-\mu\epsilon^{\alpha\beta\gamma}a_\alpha^{(1)}f_{\beta\gamma}^{(2)}\,, \qquad (1.1)$$

where a "relativistic" (2+1)-dimensional notation has been used. The remainder of the Lagrangian is supposed not to contain the gauge fields $a^{(i)}$.

From the form of Eq. (1.1), particles of species 1 are charged with respect to the fictitious field $a^{(1)}$, and particles of species 2 are charged with respect to the fictitious field $a^{(2)}$. The equations of motion obtained by varying the Lagrangian with respect to the vector potentials $a_\alpha^{(i)}$ read

$$qj^{(1)\alpha}=\mu\epsilon^{\alpha\beta\gamma}f_{\beta\gamma}^{(2)},\quad qj^{(2)\alpha}=\mu\epsilon^{\alpha\beta\gamma}f_{\beta\gamma}^{(1)}\,. \qquad (1.2)$$

From these equations, a number of things are immediately clear. First, upon looking at the $\alpha=0$ components one sees that this construction does implement the idea mentioned above—i.e., particles of each species carry flux seen by the other. Second, it is evident that consistency requires that the $j^{(r)}$ are conserved currents. Third, one sees the symmetry of the construction between the two species. [Indeed an integration by parts changes the last, Chern-Simons term of (1.1) into something of the same form but with $1\leftrightarrow2$.] This symmetry is closely tied up with the Galilean invariance of the theory. (The additional terms actually have a much wider invariance—they can be defined even in the absence of a metric, and are generally covariant.) Finally, it is clear from Eq. (1.2) that the gauge fields $a^{(1)},a^{(2)}$ really are fictitious. For the Lagrangian is gauge invariant, and the gauge invariant content of an Abelian gauge field is encoded in its field strength f (at least locally); but Eq. (1.2) shows that the field strengths (and not merely, as in Maxwell electrodynamics, their gradients in space and time) are completely determined by the particle currents.

At a microscopic level, the effect of a flux tube on a charged particle is to add a phase factor proportional to the winding multiplying the amplitude for trajectories where these objects wind around each other. This alters the spectrum of allowed relative angular momenta. Whereas ordinarily the relative orbital angular momentum between two particles is fixed to be an integer multiple of $\hbar$, in the presence of the interaction equation (1.2) the spectrum of allowed angular momenta is shifted by $(q^2/\mu)\hbar$. This effect is similar to the effect of quantum statistics for identical particles, with the only difference being that in the case of identical particles the quantization is in steps of $2\hbar$. Thus it seems sensible to refer to the *mutual statistics* of the different particles, and to parametrize it by the angle $\theta/\pi=q^2/\mu$.

(2) Layered Hall media.—Trial wave functions of the form

$$\Psi^{(m_1,m_2,n)}(w_i,z_i)=\Pi(w_i-w_j)^{m_1}\Pi(z_i-z_j)^{m_2}\Pi(w_i-z_j)^n \times\exp[-\tfrac{1}{4}eB(\Sigma|w_i|^2+\Sigma|z_i|^2)] \qquad (2.1)$$

have been suggested [2] to describe incompressible liquid quantized Hall states in situations where two distinct kinds of electrons are relevant. (Actually, only the case $m_1=m_2$ seems to have been considered previously.) The distinction might be between electrons of different spin, or between electrons having distinct wave functions in the transverse direction, such as electrons in two separate layers. Here the w_i and z_i are labels for the two-dimensional position of the different sorts of electrons. If m_1 and m_2 are both odd this is an acceptable wave function as far as quantum statistics is concerned, since it is antisymmetric between the identical fermions.

Let the number of w electrons be N_1 and the number of z electrons be N_2, both large. Then the wave function of Eq. (2.1) will describe a sensible droplet, with both species extending out to the same radius, if the degree of the polynomial in Eq. (2.1) is the same for each variable. This condition can also be derived most elegantly in the formalism where the state is defined on the surface of a sphere, where it is the condition that each electron sees the same charge for the magnetic monopole at the center. Thus we must require

$$N_1m_1+N_2n=N_1n+N_2m_2\,. \qquad (2.2)$$

If this requirement is satisfied, it is not difficult to calculate that the total filling fraction (adding electrons in both layers) is

$$\nu=(m_1+m_2-2n)/(m_1m_2-n^2)\,. \tag{2.3}$$

Based on experience with the Laughlin theory of single-species Hall states, and some numerical work, it does not seem absurd to consider the universality class of wave functions having correlations of the sort represented by Eq. (2.1) as candidates to describe incompressible quantum liquids at these filling fractions, especially for layered systems. In fact, a recent experiment [3] may be plausibly interpreted as the discovery of the $(m_1,m_2,n)=(3,3,1)$ state. It will be most interesting to explore the possible existence of incompressible states with $m_1\neq m_2$ experimentally. They exist only for definite, locked values of the filling fractions for *both* layers simultaneously. Thus, for example, (5,3,1) requires total filling fraction $\nu=\frac{3}{7}$, with $\nu_1=\frac{1}{7}$ and $\nu_2=\frac{2}{7}$.

Actually the exact definition of precisely what the universal correlations implicit in the states represented by the trial wave function of Eq. (2.1) are is somewhat elusive. Some characterizations which have proved effective for traditional quantized Hall states are in terms of topological quantum numbers of the states on higher genus Riemann surfaces [4], quantum numbers of edge states for bounded samples [5], and quantum numbers (charge and statistics) of quasiparticles [6]. Here I shall briefly discuss the first and third of these characterizations for the new states.

Probably the two most useful topological numbers associated with quantized Hall states on higher genus Riemann surfaces, which can provide signatures for them in numerical experiments, are the particle-number–flux displacement on a sphere and the degeneracy on a torus. The particle-number–flux relation on a sphere can be derived simply by doing the calculation of the filling fraction more carefully for finite N_i. The result is

$$(m_1m_2-n^2)N-(m_1+m_2-2n)N_\phi=2m_1m_2-(m_1+m_2)n \tag{2.4}$$

for the total number and

$$N_j=\frac{1}{m_1m_2-n^2}[(m_{j'}-n)N_\phi+(m_j-n)m_{j'}] \tag{2.5}$$

for the partial numbers, where j' is the index unequal to j. In general, the displacements from the naive filling fractions are sizable, and demanding that the N_j be integers severely restricts the possible values of N_ϕ. Thus, for example, the (5,3,1) state is potentially visible for $N_\phi=8$ with $N_1=N_2=2$ or for $N_\phi=15$ with $N_1=3$, $N_2=5$.

On a sphere the ground state is nondegenerate but the particle number is displaced from its limiting ratio to the flux; on a torus the reverse is true. The degeneracy of the states under discussion on a torus is $m_1m_2-n^2$. This may be inferred from the statistics of the quasiparticles as will appear below, following the arguments of [7] or from the description of the ground state in terms of an effective pure Chern-Simons theory [8]. The relevant Chern-Simons theory is a U(1)×U(1) theory with the action

$$\mathcal{L}=\frac{1}{2\pi}n_{ab}\int\epsilon^{\alpha\beta\gamma}a_\alpha^{(a)}f_{\beta\gamma}^{(b)}\,, \tag{2.6}$$

where $n_{11}=m_1$, $n_{22}=m_2$, $n_{12}=n_{21}=n$, which is immediately suggested by the form of Eq. (2.1).

The charged quasiholes are of two types. One may construct them, as proposed by Laughlin [9] for single-layer quantum Hall states, by inserting fictitious flux tubes in either of the two layers. Thus the first type is constructed as

$$\Psi^{(m_1,m_2,n)}(w_0,w_i,z_i)=\Pi(w_i-w_0)\Psi^{(m_1,m_2,n)}(w_i,z_i)\,. \tag{2.7}$$

This construction inserts a quasihole in the w layer at w_0; there is of course a corresponding construction for quasiholes in the z layer and closely related constructions for the quasiparticles.

It is a fairly straightforward matter to adapt the technique of Arovas, Schrieffer, and Wilczek [6] to calculate the quantum numbers of these excitations. One finds that the quasiholes in the first layer are anyons with charge $(m_2-n)/\Delta$ and statistical parameter

$$\theta_{11}/\pi=m_2/\Delta\,; \tag{2.8}$$

similarly, of course, in the second layer we have charge $(m_1-n)/\Delta$ and

$$\theta_{22}/\pi=m_1/\Delta\,, \tag{2.9}$$

where $\Delta\equiv m_1m_2-n^2$. In addition there is nontrivial *mutual statistics* between the quasiholes in different layers, with

$$\theta_{12}/\pi=-n/\Delta\,. \tag{2.10}$$

It is no accident that the entries of the statistical matrix form the inverse of the matrix n that appears in Eq. (2.6)—that effective Lagrangian encodes the number of fictitious flux tubes to be attached to electrons of the different species, while the statistical matrix (roughly speaking) encodes what fraction of an electron a unit flux tube represents.

(3) Superconductivity mechanism.—The fundamental concepts underlying the anyon mechanism of superconductivity [10] seem extremely powerful and appealing, and there have been many papers exploring their possible relevance to the high-temperature superconductivity of the copper oxides [11]. It has generally been supposed that the mechanism inevitably involves violation of the discrete symmetries P and T. Unfortunately despite some early encouraging indications [12] recent experiments [13] have cast considerable doubt upon the hy-

pothesis of P and T breaking in the copper oxides, and thus have cast a shadow over the whole circle of ideas.

Roughly speaking, the basic mechanism of anyon superconductivity can be described as follows. Anyons with statistical parameter $\theta=\pi(1-1/n)$ can be represented approximately as fermions—i.e., particles with statistical parameter $\theta=\pi$—moving in a fictitious average magnetic field of magnitude

$$b_{\text{avg}}=(2\pi/q)\rho_{\text{avg}} \tag{3.1}$$

together with residual *short-range* interactions. Here q is the fictitious charge of each anyon. Equation (3.1) is an immediate consequence of representing the anyons as fermions carrying fictitious charge and flux, upon replacing the pointlike flux fixed to the particles by its uniform average. This average field approximation becomes arbitrarily good at long wavelengths as $n\to\infty$, and appears to be good qualitatively even for small n. Now for n integer the fermions will exactly fill n Landau levels, and there will be a gap in the charged particle spectrum. On the other hand, unlike what would happen for fermions in an ordinary external magnetic field the fluid remains compressible: Because b_{avg} is pinned to ρ_{avg}, slow variations of ρ will not require exciting particles across the Landau-level gap, and can be made with a small cost in energy. Thus there is a single low-energy mode, which is necessarily dissipationless: This is the superflow mode. If the anyons are electrically charged, this mode will produce not superfluidity but superconductivity, according to the London-Landau-Ginzburg-Anderson-Higgs mechanism.

As described, this mechanism for superfluidity clearly relies heavily on the dynamical potency of the fictitious magnetic field. Such a magnetic field violates the discrete symmetries P and T.

It has been appreciated for a long time that one can have anyons without violating the discrete symmetries, by a doubling procedure [14]. That is, one may consider a theory in which for every species with θ statistics there is also a species with $-\theta$ statistics. Then (if the couplings are otherwise symmetrical) it will be possible to combine the naive parity and time reflections with interchange of species type, to construct a valid symmetry. In this way one can readily construct models with two independent anyon superfluids, each coupled to its own fictitious gauge field, that respect the discrete symmetries. However, these models feature an additional "leftover" superfluid mode, which presumably precludes their use for describing presently known superconductors. The additional mode consists of density modulations where the two anyon fluids are 180° out of phase.

I would now like to consider a simple new possibility opened up by disassembling anyons. Consider two species of mutually distinguishable fermions, described by densities and currents $\rho^{(r)}$ and $j^{(r)}$ where $r=1,2$. Suppose that their mutual statistics is π/n, implemented by the Lagrangian of Eq. (1.1) with $q^2/\mu=2\pi/n$. To get a qualitative indication of the effects of the interactions, let us again follow the procedure of replacing the magnetic fields pinned to the particles by their uniform average. Then if the densities of the two fluids are equal, exactly n Landau levels will be filled for each. For these values of the parameters, the arguments for anyon superfluidity will proceed essentially as before. The densities of the two fluids may oscillate together without upsetting the conditions for filled Landau levels; but out-of-phase oscillations will upset these conditions, and will face an energy gap.

The analogous arguments for anyon superfluidity can be verified by controlled calculations in the limit of large n, and although no thorough analysis has been done I fully expect that will be true in the present context too. For $n=2$ the mutual angular momenta will be quantized to be half-odd integer. This quantization condition does *not* violate P or T symmetry, because the spectrum is unchanged if we change the sign of the angular momenta.

With the identification of the two fermion fluids as spin-up and spin-down electrons, the model considered here may be considered as a possible realization of Anderson's ideas regarding statistical repulsion between opposite spin electrons [15]. However, the mechanism presented here, in contrast to Anderson's ideas as I understand them, is intrinsically two dimensional and contains commensurability effects—especially, the anyon superconductivity mechanism— that are not clearly present in his formulation. Also the precise form of interaction following from the Lagrangian of Eq. (1.1), although it has a similar characteristic divergence in the forward direction, is different from the form he proposes. The scattering amplitude for momenta k,k' is

$$V_{k,k'}\propto\frac{k\times k'}{|k-k'|^2}, \tag{3.2}$$

as opposed to a multiple of $(\epsilon_k-\epsilon_{k'})/|k-k'|^2$. Also, there are additional three-body interactions.

Many variations on the specific model proposed here could of course be considered; and it remains to be seen which if any can be derived from more microscopic considerations. The major points that emerge clearly from the present discussion are that there is a valid—local, and indeed rather simple—potential universality class realizing the idea of statistical repulsion between unlike particles in two dimensions, and that in this context the mechanism of anyon superconductivity is readily implemented in essentially new models, including ones which respect the discrete symmetries P and T.

I will conclude with the following comments.

(1) Understanding the phase structure of disassembled anyon gases in general is a problem of considerable intrinsic interest. I have no substantial results to report on this problem, but I would like to remark that it has an important physical analog and that there is a plausible

general principle suggesting the existence of ordered phases. On a sphere subject to a normal magnetic field, the single-electron wave functions in the lowest Landau level are polynomials of a fixed degree—thus, each is associated with a definite number of zeros. These zeros may be regarded as flux tubes for a fictitious gauge field seen by the electron. (Note that in the lowest Landau level the relative angular momentum has a definite sign, so that in this context changing the spectrum by an integer unit is meaningful, unlike for free particles.) In the quantized Hall states these effective flux tubes are bound to the other electrons, as is clearly indicated by Laughlin's wave function. In the normal state, of course, they dissociate. A gas of particles with mutual statistics can have related transitions, wherein flux tubes and particles bind or unbind. It seems reasonable to anticipate that in general the formation of effective bosons, with subsequent boson condensation, would be energetically favored at low temperature. Thus, for example, two gases of $\theta/\pi=3/8$ anyons with mutual statistics $\theta/\pi=-3/8$ might be expected to condense into hybrid pairs. The transition discussed above from two fermion gases with mutual half-fermion (semion) statistics to a superfluid of correlated pairs (pairs of the first species and pairs of the second species, not hybrids), forming effective bosons, is a different embodiment of the same principle.

(2) In three spatial dimensions one probably ought not to expect a continuous range of possibilities for mutual statistics, any more than for ordinary statistics, since angular momentum, to which these are closely related, is intrinsically quantized. However, there is a sort of analog for mutual statistics in three dimensions. Ordinarily the relative angular momentum between two different species of particles is integral. However, it is a famous fact [16] that the orbital angular moment between minimal magnetic monopoles and electric charges is half-odd integral; thus we may say such particles have nontrivial mutual statistics. This line of thought suggests a question. For identical particles the angular momentum is quantized in steps of $2\hbar$—odd for fermions, even for bosons. At the kinematic level one could certainly imagine displacing these rules by $\frac{1}{2}\hbar$. Is it possible to implement this possibility in a local field theory?

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