Quark-Antiquark Regge Trajectories in Large-N_c QCD

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We apply methods developed by Lovelace, Lipatov, and Kirschner to evaluate the leading Regge trajectories $\alpha(t)$ with the quantum numbers of nonexotic quark-antiquark mesons at $N_c=\infty$ in the limit $t\rightarrow -\infty$ where renormalization-group-improved perturbation theory should be valid. We discuss the compatibility of nonlinear trajectories with narrow-resonance approximations.

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It is unlikely that quantum chromodynamics, the consensus theory of strong interactions, can be exactly solved with realistic values for all parameters. However, asymptotic freedom allows the application of weak-coupling techniques such as perturbation theory to obtain the predictions of QCD for processes controlled by short-distance dynamics. Besides high-momentum-transfer collision phenomena, one can hope to use such weak-coupling techniques for computing the mass spectrum of hadrons containing only very heavy quarks. But for hadrons containing light quarks and also for glueballs, strong-coupling dynamics is unavoidable. Even idealizing the light quark masses to zero does not simplify the dynamics enough for an analytical treatment: The S matrix is nontrivial in all channels including those with particle production, and the bound-state spectrum includes all nuclei as well as the lowest mass hadron in each flavor sector.

That is why 't Hooft's idea of exploiting the $N_c\rightarrow\infty$ limit [1] is so attractive. In this limit the scattering amplitudes involving hadrons vanish and in lowest nonvanishing order are meromorphic in the channel invariants, just as the tree approximation to a quantum field theory. Nor do the nuclei bind in this limit. Thus an exact solution in this limit really would be significantly simpler than the exact solution at $N_c=3$. Unfortunately, with currently available methods the infinite-N_c theory seems almost as intractable as the finite-N_c theory. Lacking a complete solution of large-N_c QCD, we think it is worthwhile to develop as much insight into the nature of the hoped-for solution as possible.

String theory started as an effort to build exactly the sort of approximation to strong-interaction dynamics that is provided by large-N_c QCD. Since that approach led to the "wrong" answer, it is important to understand how the expected properties of QCD are different from those of string theory. In string theory the Regge trajectory functions $\alpha_{\rm string}(t)=\alpha' t+\alpha_0$, where $\alpha'=1/2\pi T_0$ with T_0 the rest tension in the string, play a central role in the string scattering amplitudes: They appear directly as the arguments of the Euler beta function which gives the lowest-order Veneziano four-string function $A_4(s,t)=g^2B(-\alpha(s),-\alpha(t))$. The meromorphy of A_4 in s and t follows directly from that of the beta function and the exact linearity of the Regge trajectories. Since the large-N_c hadron amplitudes are also expected to be meromorphic in s and t, the trajectory functions themselves should be good signatures of the similarities and differences between string theory and QCD. Also they might carry some hints about the solution of large-N_c QCD.

In this Letter we study the Regge trajectories of large-N_c QCD in the "meson" channels (i.e., those interpolating the rotational states of quark-antiquark mesons), in the limit of large negative t where perturbative QCD should be applicable. We follow ideas and methods developed in Refs. [2-4]. These are essentially renormalization-group-improved calculations based on summing leading logarithmic contributions of Feynman graphs. Such methods can only give the trajectory functions in the weak-coupling approximation. Since the coupling $\lambda\equiv N_c g_s^2/4\pi^2$ "runs" with the scale, $\lambda(-t)\sim 12/11\ln(-t/\Lambda_{\rm QCD}^2)$ for $t\rightarrow -\infty$, this means that we can obtain only the large-negative-t behavior of the Regge trajectories using these methods. [Note that the large-positive-t behavior of the trajectories is characterized by the confining force and should be asymptotically linear, $\alpha_{\rm QCD}(t)\sim t/2\pi k$, where kR is the confining term in the $q\bar{q}$ interaction energy.]

Our first task is to identify the leading logarithmic contributions to a scattering process involving the exchange of a $q\bar{q}$ pair. In any gauge theory the ladder diagrams, which iterate gauge-boson exchange between two fermion lines, contribute two powers of $\ln(s/\mu^2)$ for each additional rung. Thus the leading logarithms are actually doubly logarithmic and dominate the single logarithms of renormalization: The leading logarithmic sums will therefore *not* include running coupling effects. Thus we proceed in two steps. First we evaluate the amplitudes to double-logarithmic accuracy and then incorporate renormalization effects which make the coupling run in the second step. As shown in 1967 [5] for QED, the first step typically leads to a fixed square root branch point in the angular momentum plane. For QED processes involving the exchange of total zero charge, the leading double logarithms come only from the ladder sum, which produces a branch point located at $J=\sqrt{2\alpha/\pi}$ where $\alpha\approx 1/137$ is the fine structure constant. When nonzero charge is exchanged, there are additional double-logarithmic contributions coming from soft "bremsstrahlung" photons which either form crossed rungs in the ladder or Sudakov vertex corrections [5]. Similarly, in QCD the double logarithmic contributions of the basic ladder diagrams of Fig. 1 are supplemented by soft gluon bremsstrahlung graphs [6]. Fortunately for us, these additional diagrams are nonleading in the $1/N_c$ expansion when the ladder structure is "hooked on" to color-singlet hadron vertices (note that these hadron "form factors," represented by the left- and right-hand parts of Fig. 1, involve on-shell mesons and cannot be calculated perturbatively); so, they do not enter into our calculation of the Regge trajectories of large-N_c QCD. Thus the double-logarithmic sum is identical to the QED zero-charge exchange case with $\lambda \equiv N_c g_s^2/4\pi^2$ substituted for $2\alpha/\pi$.

Since renormalization effects are neglected in the leading double-logarithmic approximation, the results depend on a fixed coupling constant λ. Asymptotic freedom must at least make the location of the singularities in the angular momentum plane vary with t according to the replacement $\lambda \to \lambda(-t)$, but actually the cut is expected to be replaced by a distribution of Regge poles accumulating at 0 as $t \to -\infty$. This phenomenon was first uncovered by Lovelace [2] for the case of ϕ^3 theory in six space-time dimensions where the accumulation point is at $J=-1$. He analyzed the Bethe-Salpeter (B-S) equation with a kernel improved to include the effects of asymptotic freedom. This equation produced partial-wave amplitudes with only pole singularities in the angular momentum plane. Since he only considered the case $t=0$, his results for the pole locations (Regge intercepts) were untrustworthy. (The low-momentum theory is a strong-coupling one and the B-S equation is not valid there.) This shortcoming was removed by Kirschner and Lipatov [4], who incorporated t dependence in leading order and

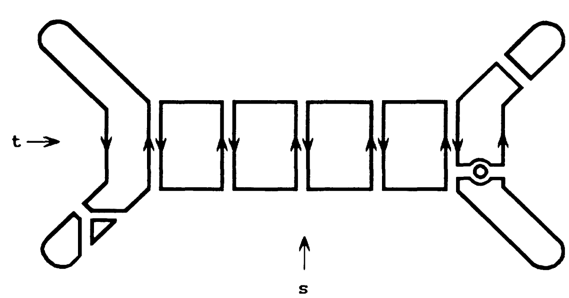

FIG. 1. A typical large-N_c diagram contributing to meson scattering with the exchange of a $q\bar{q}$ ladder structure. The leading log approximation as $s \to \infty$ is the sum of graphs with an arbitrary number of gluon rungs represented by the vertical double lines. In the double-line representation of Feynman diagrams developed in Ref. [1], the gluon propagator $\langle A_\mu(x)_i^k \times A_\nu(0)_j^l \rangle$ carries the color factor $\delta_i^l \delta_j^k$; each Kronecker delta is represented by one of the double lines. Thus each closed line in this diagram supplies a factor $\sum_{i=1}^{N_c} \delta_i^i = N_c$.

obtained $\alpha(t)$ for large negative t instead of $\alpha(0)$. With large enough t, the effective coupling is weak, justifying the B-S equation. Earlier, Lipatov [3] had obtained similar results for the asymptotic behavior of the Pomeron (glueball) trajectory in QCD. In this note we find the corresponding asymptotic behavior of the $q\bar{q}$ trajectories in large-N_c QCD.

In order to incorporate the running coupling, we consider the B-S equation which sums the ladder subgraphs in Fig. 1. We represent the Green's function for the ladder subgraphs as a matrix Ψ_{ab} in the Dirac indices of the $q\bar{q}$ lines coming in at the left. We apply the Dirac operators to these two lines in the coordinate representation to obtain (for simplicity we take all $m_q=0$)

$$\gamma\cdot\partial_1 \Psi(x_1,x_2;y_1,y_2)\gamma\cdot\overleftarrow{\partial}_2 = -\delta(x_1-y_1)\delta(x_2-y_2) + \lambda(x_{12}^{-2})\gamma^\mu \Psi \gamma^\nu d_{\mu\nu}(x_{12}),$$

where we have followed Lovelace's treatment of ϕ_6^3, replacing the coupling constant by the running coupling $\lambda(x^{-2}) \approx -12/11\ln(x^2\Lambda_{\rm QCD}^2)$. In a general covariant gauge we define the coordinate space propagator by $d_{\mu\nu}(x)/4\pi^2$ with

$$d_{\mu\nu}(x) = -i\int \frac{d^4p}{(2\pi)^2} e^{ix\cdot p} \frac{\eta_{\mu\nu} - \zeta p_\mu p_\nu/p^2}{p^2 - i\epsilon} = (1+\zeta)\eta_{\mu\nu}/2x^2 + (1-\zeta)x_\mu x_\nu/x^4 .$$

The B-S equation is not gauge invariant, but violations of gauge invariance will be small for weak coupling. Thus if we only keep leading-order answers, our results should be gauge invariant. We keep ζ arbitrary so we can confirm this. We expect this equation to be accurate when $\lambda \ll 1$, i.e., for $x_{12}^2\Lambda_{\rm QCD}^2 \ll 1$. The singularities in the t channel are controlled by the solutions of the homogeneous equation

$$\gamma\cdot\partial_1 \Psi \gamma\cdot\overleftarrow{\partial}_2 = \lambda(x_{12}^{-2})\gamma^\mu \Psi \gamma^\nu d_{\mu\nu}(x_{12}) .$$

It is convenient to work with c.m. and relative coordinates $r=(x_1+x_2)/2$ and $\rho = x_1 - x_2$ and to Fourier transform with respect to r, whose conjugate variable is q so that $q^2=-t$. Then the homogeneous equation reads

$$\gamma\cdot(\partial/\partial\rho - iq/2)\tilde{\Psi}(\rho,q)\gamma\cdot(-\overleftarrow{\partial}/\partial\rho - iq/2) = \lambda(\rho^{-2})\gamma^\mu\tilde{\Psi}(\rho,q)\gamma^\nu d_{\mu\nu}(\rho) .$$

We should only use this equation for small ρ and large q when the effective coupling associated with both scales in the problem is small.

For $q\rho \ll 1$, but $q,\rho^{-1} \gg \Lambda_{\rm QCD}$, the B-S equation becomes quite manageable. This limit reduces it to the $q=0$ case, with its O(4) symmetry. As in Ref. [4] one can conveniently consider the lth partial waves in this limit by making the ansatz

$$\tilde{\Psi}_l(\rho) = \frac{(\rho\cdot\xi)^{l-1}}{|\rho|^l}\left[\frac{\rho\cdot\gamma\rho\cdot\xi}{\rho^2} f(|\rho|) + \xi\cdot\gamma g(|\rho|)\right],$$

with ξ^μ a fixed lightlike four-vector, so that one is forming traceless symmetric tensors of rank l. Plugging this ansatz into the B-S equation then yields the pair of equations

$$[(\mathcal{D}-l)^2-4]f+2[\mathcal{D}-l][\mathcal{D}-l-2]g = \lambda(\rho^{-2})[\zeta f-(1-\zeta)g]\,, \quad (1)$$

$$2l[\mathcal{D}+1]f-[\mathcal{D}-l]^2 g=\lambda(\rho^{-2})g\,.$$

Here $\mathcal{D}\equiv|\rho|\partial/\partial|\rho| = \partial/\partial\ln(|\rho|\Lambda)$ is simply the scaling derivative in the magnitude of ρ.

Equation (1) is a spinor version of the small-ρ B-S equation for the ϕ_6^3 theory analyzed in Refs. [2,4]. With $\lambda(\rho^{-2})$ approximated by $-6/11\ln(|\rho|\Lambda)$, the Laplace transforms of $\lambda f,\lambda g$ satisfy a pair of first-order differential equations in $\mathcal{D}\equiv -2i\hat{v}$. Unlike the single equation in the ϕ_6^3 case which can immediately be integrated, this pair of equations is equivalent to a single-component second-order equation which cannot be so readily solved. However, the leading angular momentum singularity (with largest Rel) is controlled (for small λ) by l and $\hat{v}$ small compared to unity. To see this note that the determinant of the coefficient matrix on the left-hand side of (1) is $-[l^2+4v^2][(l+2)^2+4v^2]$ in v space. For small l,v, the second of (1) shows that $2lf\approx\{\lambda(\rho^{-2})+[\mathcal{D}-l]^2\}g$. Inserting this approximate form for f into the first equation and making the same approximations there gives $(-\mathcal{D}^2+l^2)g\approx\lambda(\rho^{-2})g$, the Laplace transform of which gives a first-order equation in v. Thus we see that this leading Regge singularity is controlled by equations independent of the gauge ζ, as we anticipated for weak coupling. Looking back to the full gauge-dependent small-ρ equations, we notice that in Landau gauge ($\zeta=0$) one can eliminate f in favor of g in a ρ-independent way. Thus in this gauge, which we choose in the following, the Laplace-transformed equations are first order in v and can be directly integrated.

Setting $\zeta=0$, solving the second equation for f, and substituting in the first we obtain

$$[l^2+4\hat{v}^2][(l+2)^2+4\hat{v}^2]g=[4+4\hat{v}^2-l^2-2l]\lambda(\rho^{-2})g\,. \quad (2)$$

The equation for g is now quite similar to the ϕ_6^3 case and we can repeat the steps in Ref. [4] to derive the asymptotic behavior of the Regge trajectories. First, (2) is solved for λg by noting that $[\hat{v},-\ln(\rho^2\Lambda^2)]=-i$ so that $R\equiv-\ln(\rho^2\Lambda^2)$ can be replaced by $i\partial/\partial\hat{v}$. Integrating the resulting equation and transforming back to the coordinate representation yields

$$\lambda(\rho^{-2})g=\int_{-\infty}^{\infty}dv\,e^{ivR-12i\Phi(v)/11}\,, \quad (3)$$

where

$$\Phi(v)\equiv\int_0^v dv'\frac{4+4v'^2-l^2-2l}{[l^2+4v'^2][(l+2)^2+4v'^2]}\,.$$

This solution of the small-ρ B-S equation is, in fact, a solution of the full B-S equation for $q=0$. But of course the B-S equation is only a good approximation for large q and small ρ. In Ref. [4] the analogous solution for the ϕ_6^3 theory is used to gain information about the large-q behavior of the trajectories by noting that for $\rho q\ll 1$ but ρ not too small one can have $\lambda(\rho^{-2})\approx\lambda(q^2)$ for a large range of ρ (essentially because the scale dependence of λ is only logarithmic). Thus instead of requiring regularity of the solution at $R=0$ as in Ref. [2], the solution is matched to that of the B-S equation with a ρ-independent coupling taken to be $\lambda(q^2)\ll 1$. For consistency of the weak-coupling approximation this matching must be imposed at large R (small ρ).

The large-R behavior of (3) is exponentially damped for extremely large R but there is oscillatory behavior for R not too large. This can be extracted by finding the saddle points v_0 which are solutions of

$$16v_0^4+8[l^2+2l+2-6/11R]v_0^2 +l^2(l+2)^2-12[4-l^2-2l]/11R=0\,. \quad (4)$$

We see that there are two real values of v_0 provided $R<\frac{12}{11}\{[4-l(l+2)]/l^2(l+2)^2\}$, which is consistent with large R provided $|l|\ll 1$. In this regime, λg is well approximated by the saddle-point evaluation

$$\lambda g\approx 2\left[\frac{6\pi}{11\Phi''(v_0)}\right]^{1/2}\cos\left[\frac{\pi}{4}+v_0R-\frac{12}{11}\Phi(v_0)\right]\,, \quad (5)$$

where v_0 is the positive solution of (4). For sufficiently large R, the R dependence of v_0 can be neglected and (5) can be matched to the small-ρ solution of the B-S equation with constant coupling $\lambda(q^2)$. Scale invariance of the finite-q B-S equation implies that the solution is a function of $\rho|q|$. In the small-ρ limit it therefore has the behavior

$$[(\rho|q|)^{-2iv_1}+(\rho|q|)^{+2iv_1}e^{i\delta(l,v_1)}] =2e^{i\delta/2}\cos[\delta/2+v_1\ln|\rho|^2|q|^2]\,, \quad (6)$$

where v_1 is v_0 with $12/11R$ replaced by $\lambda(q^2)$.

Replacing v_0 by v_1 in (5) and comparing to (6) gives the matching condition

$$\lambda(q^2)\Phi(v_1)-v_1=\frac{1}{\ln(q^2/\Lambda^2)}\left\{r\pi+\frac{\delta}{2}+\frac{\pi}{4}\right\}\,, \quad (7)$$

where r=integer. We shall find that the consistent small-coupling solution of these equations gives $l=O(\lambda^{1/2})$ and $v_1=O(\lambda^{2/3})$. Thus $v_1/l^2=O(\lambda^{-1/3})$ and $v^3/l^4\sim 1$. Neglecting all terms that vanish at zero λ gives $\Phi(v_1)\approx v_1/l^2-4v_1^3/3l^4$. With these approximations $\lambda\approx l^2(1+4v_1^2/l^2)$, so the leading term on the left-hand side of Eq. (7) cancels and we are left with

$$\frac{8v_1^3}{3l^2}\approx\frac{1}{\ln(q^2/\Lambda^2)}\left\{r\pi+\frac{\delta}{2}+\frac{\pi}{4}\right\}.$$

Replacing l^2 by $\lambda(q^2)$, we thus find the following behavior for the Regge trajectories $\alpha_r(t) \sim [\lambda(-t) - 4\nu_r^2]^{1/2}$ as $t \to -\infty$,

$$\alpha_r(t) \underset{t\to-\infty}{\sim} \lambda^{1/2}(-t) - 2\lambda^{5/6}(-t)\left[\frac{11}{32}\left(r\pi + \frac{\delta}{2} + \frac{\pi}{4}\right)\right]^{2/3} + \cdots .$$

Notice that there are an infinite number of trajectories accumulating at 0 in the limit $t \to -\infty$, with almost identical behavior of those of the ϕ_6^3 theory obtained in Ref. [4].

The phase δ is not determined from the small-ρ dynamics considered so far. It must be determined by the dynamics at $\rho q \sim 1$. However, except in exceptional cases, $\delta = \pi$ in the limit we are considering. This is because this limit involves $\nu \approx 0$ in (6). For $\nu = 0$ the two behaviors $\rho^{\pm 2i\nu}$ are replaced by 1 and $\ln\rho$. Generically, both behaviors will be present, and unless the coefficient of $\ln\rho$ exactly vanishes, the behavior for ν slightly different from zero must be

$$\frac{a}{\nu}[(q\rho)^{-2i\nu} - (q\rho)^{2i\nu}] + b(q\rho)^{-2i\nu} + c(q\rho)^{+2i\nu}$$

with a,b,c finite at $\nu = 0$. If the $\ln\rho$ term is present at $\nu = 0$, then $a \neq 0$ there and $\delta = \pi$ at $\nu = 0$. This is analogous to the generic vanishing of phase shifts at zero energy. In that analogy the case $a = 0$ corresponds to a "zero energy resonance." For the ϕ_6^3 case, $\delta = \pi$ was shown by explicit solution of the constant-coupling B-S equation using conformal invariance [4]. In gauge theories, a conformal transformation changes the gauge condition, so the B-S equation, being gauge noninvariant, is scale invariant but not conformally covariant. Lacking an explicit solution, we can only state that it is likely, but not proven, that $\delta = \pi$ for large-N_c QCD.

We close with some comments about the significance of nonlinear Regge trajectories for large-N_c QCD. There is a common belief [7] that narrow-resonance approximations require exactly linear (or at worst polynomial) Regge trajectories. However, this conclusion depends on a maximal analyticity assumption that the trajectory functions are free of singularities in the t plane cut on the right at threshold branch points [8]. Since $\mathrm{Im}\,\alpha(t)$, the discontinuity across the threshold cut, is proportional to the resonance widths, the trajectories would then be entire functions in the limit of zero-width resonances. We have seen that the $q\bar{q}$ trajectories of large-N_c QCD approach constants as $t \to -\infty$, and confinement together with infinite N_c implies linear behavior as $t \to +\infty$ as well as no threshold branch points. The inescapable conclusion is that the maximal analyticity assumption fails for the Regge trajectories of infinite-N_c QCD and there are additional singularities in the t plane. This is probably also true at $N_c = 3$ since there is no good physical basis for the absence of additional singularities. In Ref. [9] (for earlier models see also Refs. [10,11]) one of us discussed some examples of narrow-resonance models with nonlinear trajectories with algebraic branch points in the complex t plane. Some of these models have trajectories which are asymptotically linear for large positive t and approach constants at large negative t. Unfortunately, they approach these constants as an inverse power of t rather than an inverse power of $\ln(-t)$, so they are not candidates for large-N_c QCD. Nonetheless, they do show that nonlinear trajectories are compatible with narrow resonances, and indicate a direction toward solving large-N_c QCD.

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