Distribution of Eigenmodes in a Superconducting Stadium Billiard with Chaotic Dynamics

H.-D. Gräf,[1] H. L. Harney,[2] H. Lengeler,[3] C. H. Lewenkopf,[2] C. Rangacharyulu,[4] A. Richter,[1] P. Schardt,[1] and H. A. Weidenmüller[2]

[1]*Institut für Kernphysik, Technische Hochschule Darmstadt, W-6100 Darmstadt, Germany*
[2]*Max-Planck-Institut für Kernphysik, W-6900 Heidelberg, Germany*
[3]*AT-Division, CERN, CH-1211 Geneva 23, Switzerland*
[4]*Department of Physics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada S7N 0W0*
(Received 8 June 1992)

The complete sequence of 1060 eigenmodes with frequencies between 0.75 and 17.5 GHz of a quasi-two-dimensional superconducting microwave resonator shaped like a quarter of a stadium billiard with a Q value of $Q \approx 10^5$–10^7 was measured for the first time. The semiclassical analysis is in good agreement with the experimental data, and provides a new scheme for the statistical analysis and comparison with predictions based on the Gaussian orthogonal ensemble.

PACS numbers: 05.45.+b

Quantum manifestations of classical chaos in systems with few degrees of freedom can be studied experimentally. In sufficiently flat microwave resonators, Maxwell's equations reduce to the Schrödinger equation for the free particle, and the condition of classical chaos is realized by properly shaping the cavity [1]. In this Letter, we report on first measurements using a superconducting cavity. The high-quality factor $Q = f/\Delta f$ (with f the frequency) of $Q \approx 10^5$–10^7 of this device allows us to measure for the first time the *complete* spectrum below 17.5 GHz; we find about 160 more eigenvalues than a similar experiment at room temperature. We employ the semiclassical approach and calculate for the first time locations and strengths of the peaks in the Fourier-transformed spectrum in terms of the shortest unstable classical periodic orbits. We find good agreement with the data. We also perform a statistical analysis of our spectrum; such an analysis is meaningful only if the spectrum is complete (no missing levels). We compare our results to those of the Gaussian orthogonal ensemble (GOE), the standard stochastic model for time-reversal invariant systems with classically chaotic dynamics, and find deviations which are successfully explained and quantitatively described by the semiclassical approach.

Two-dimensional billiards are among the most thoroughly studied models for classical chaos and its quantum manifestations [2]. The transition from integrable to nonintegrable classical behavior can be studied by changing the shape of the billiard. On the quantum level, this transition is accompanied by a crossover from Poisson statistics to GOE statistics in the semiclassical region. The properties of eigenfunctions and problems of chaotic scattering have also received attention [3–11].

These theoretical studies have been put to an experimental test only very recently [1,12,13] using microwave excitation of quasi-two-dimensional cavities. The formal similarity between the stationary Schrödinger and Helmholtz equations, the latter considered for frequencies f below a critical frequency $f_{\max}$ where the electrical field distribution depends only on the planar variables x and y, renders such experiments suitable for the study of quantum phenomena. The height d of the microwave cavity in the z direction is bounded by $d < c/2f_{\max}$.

Here, we present results obtained with a superconducting niobium cavity with $Q \approx 10^5$–10^7, which has the shape of a quarter of a Bunimovich stadium billiard with inner dimensions $r = 20$ cm, $a = 36$ cm, and height $d = 0.8$ cm (see upper right-hand corner of Fig. 1), corresponding to $\gamma = a/r = 1.8$. With only a quarter of a stadium, one is restricted to a single symmetry class of the full problem [6]. The cavity has been put into one of the cryostats of the new superconducting Darmstadt electron linear accelerator S-DALINAC, where it was cooled down to 2 K together with the accelerating structures [14]. As indicated in Fig. 1, three antennas were located in small holes (3 mm diameter). To keep their influence on the field distributions negligibly small care was taken that they did not penetrate into the cavity. Three independent trans-

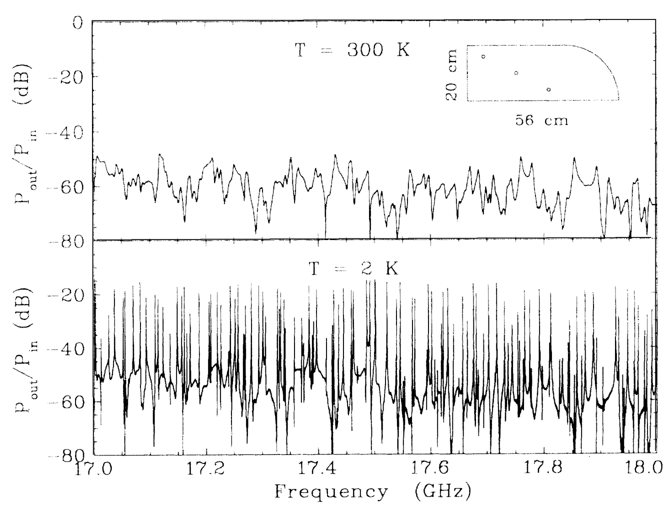

FIG. 1. Measured spectrum between 17 and 18 GHz. The upper part is taken at room temperature (normally conducting); the lower part at 2 K (superconducting). Inset: Illustration of the shape of the resonator and the positions of the antennas.

mission spectra were taken by using a vector network analyzer and different combinations of the antennas. All spectra were checked against each other for consistency and finally combined into a single spectrum. It consists of 1.8×10^6 data points in the range from 0.75 to 18.75 GHz. The step width of the measurement was 10 kHz. The large signal-to-noise ratio (>100) allows us to identify each resonance by taking many data points in the tails. This is why we are almost certain not to have missed any modes. Further support for this statement derives from the fact that the smallest observed spacing is 300 kHz. In Fig. 1, the spectra in the range from 17 to 18 GHz measured at $T=2$ and 300 K are shown for comparison. The improvement of the resolution by about 3 orders of magnitude is striking. To ensure the two dimensionality of the cavity the analysis of the spectra has been confined to $f<17.5$ GHz [1]. Up to this frequency 1060 eigenmodes were counted (compared to 898 modes at room temperature).

We analyzed these data in two ways: using the semiclassical approach and the GOE.

The semiclassical theory suggests to express the level density $\rho(k)=\sum_n\delta(k-k_n)$, where n labels the eigenmodes and k is the wave number, as a sum of a smooth part $\rho^{\rm sm}(k)$ [defined as the average of $\rho(k)$ over a k interval] and a fluctuating part $\rho^{\rm fl}(k)$ (with average zero). Each part is associated with a different physical aspect. Usually the smooth part $\rho^{\rm sm}(k)$ relates to the volume of the classical energy-allowed phase space. For billiards, an improved version is given by the Weyl formula, which also includes surface corrections

$$\rho^{\rm Weyl}(k)=\frac{A}{2\pi}k-\frac{P}{4\pi}+\cdots\,, \tag{1}$$

where A is the area of the billiard and P its perimeter. Equation (1) does not contain *any* information regarding the character of the underlying classical dynamics of the system.

According to the Gutzwiller trace formula [15], the fluctuating part of the density is given in terms of the periodic orbits, the only elements of the classical dynamics that manifestly survive quantization and are seen in the spectrum:

$$\rho^{\rm fl}(k)=\frac{1}{\pi}\,{\rm Re}\sum_\mu\frac{L_\mu}{[{\rm tr}\mathbf{M}_\mu-2]^{1/2}}\exp[ikL_\mu-i\phi_\mu]\,. \tag{2}$$

Here, the index μ labels the periodic orbits and their recurrences, $\mathbf{M}_\mu$ their monodromy matrix, L_μ their length over one period, and ϕ_μ their Maslov index, where all μ's are isolated. It is Eq. (2) that carries information regarding the chaotic (or regular) character of the classical dynamics and the instability (or stability) of trajectories.

Unfortunately the Gutzwiller trace formula is not applicable to all orbits of the stadium billiard. There is a family of nonisolated marginally stable periodic orbits that bounce between the two straight segments of the billiard. The presence of these orbits is manifested in the spectrum in two different ways: First, there is an approximate quantization rule given by $k_jr=j\pi$. The experimental data agree with this rule astonishingly well. We were able to identify all predicted k_j within a precision of $\frac{1}{1000}$, a consequence of the extraordinary resolution of the superconducting measurement. Second, there exists an additional smooth effect in $\rho(k)$ not accounted for by the Weyl formula, which can be seen in Fig. 2: The cumulative level density, $N(k)=\int_0^k dk'\rho(k')$, shows smooth periodic oscillations (with fixed period $2\pi/r$) around the value given by the Weyl formula [16]. These facts can be understood by the semiclassical analysis of the contribution of the bouncing ball orbits to the spectrum. This contribution can be evaluated [17] by considering the semiclassical propagator [see Eq. (12.21) in [15]] restricted to trajectories that never leave the rectangular sector of the billiard, $K^{\rm bb}(\mathbf{q},\mathbf{q}',t)$. The procedure is then standard: The Laplace transform of ${\rm tr}K^{\rm bb}(\mathbf{q},\mathbf{q}',t)$ gives the trace of the Green's function due to the rectangular sector, whose imaginary part is related to the spectral density. The result is

$$N^{\rm bb}(k)=\frac{1}{(2\pi)^{3/2}}\frac{a}{r}\sqrt{2kr}\sum_{m=1}^{\infty}m^{-3/2}\cos\left(2mkr-\frac{3\pi}{4}\right). \tag{3}$$

Equation (3) very nicely reproduces the experimental data as can be seen in Fig. 2. This additional smooth correction has to be added to the Weyl formula to obtain the proper $N^{\rm sm}(k)$ [or $\rho^{\rm sm}(k)$].

The effect of the isolated periodic orbits is most instructively displayed in the Fourier-transformed (FT) spectrum of $\rho^{\rm fl}(k)$,

$$\tilde\rho^{\rm fl}(x)=\int_{k_{\rm min}}^{k_{\rm max}}dk\,e^{ikx}[\rho(k)-\rho^{\rm sm}(k)]$$
$$=\sum_n e^{ik_nx}-\int_{k_{\rm min}}^{k_{\rm max}}dk\,e^{ikx}\rho^{\rm sm}(k)\,, \tag{4}$$

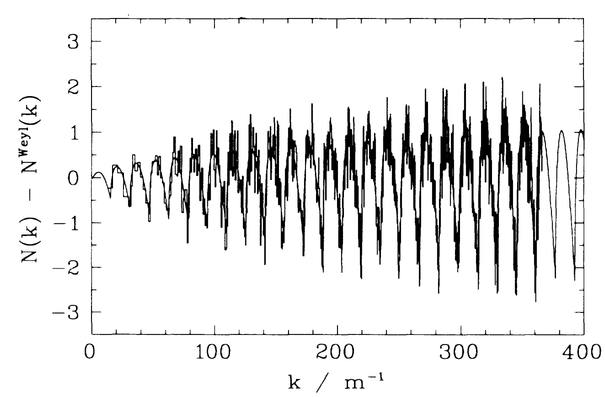

FIG. 2. The histogram gives the cumulative level density $N(k)-N^{\rm Weyl}(k)$ as a function of the wave number k. The solid line shows the semiclassical prediction of Eq. (3) for the contribution of the bouncing ball orbits. There is excellent agreement between experiment and theory. The two are only discernible at low and high wave numbers.

where $[k_{\min}, k_{\max}]$ is the wave-number interval in which data were taken. Equation (2) shows that to each isolated periodic orbit there corresponds a peak in the FT spectrum whose intensity is determined by the length and the instability of the periodic orbit. In Fig. 3, we show the power spectrum $|\tilde{\rho}^{\rm fl}(x)|^2$ of $\rho^{\rm fl}(k)$ (solid line) compared to the theoretical result (dashed line). The latter is obtained by identifying the thirty shortest periodic orbits of the system and by calculating their length and monodromy matrix [18]. We are able to reproduce most of the amplitudes up to the length of $x=2.3$ m, where we start to miss periodic orbits. For the peak at $x=1.32$ m corresponding to the so-called "whispering gallery" orbits [6] we summed the contribution of the ten most stable orbits. For this family of orbits tr**M** increases fast with the number of collisions with the circular part, so that orbits approaching the circle give a small contribution to the power spectrum. However, we fail to reproduce the amplitude at $x=1.37$ m, possibly because it is due to a slight geometry imperfection. The radius of curvature might not change abruptly, but smoothly at the transition from the straight to the circular section of the boundary, possibly yielding several periodic orbits of similar length and hence a broad big peak. For other peaks this small imprecision (if this is the correct explanation) does not imply a sizable effect. The peaks located at 1.6 and 2.0 m are remanents of the bouncing ball orbits and are suppressed by taking $\rho^{\rm fl}=\rho-\rho^{\rm Weyl}-\rho^{\rm bb}$. If $\rho^{\rm bb}$ is not subtracted the power spectrum is dominated by the bouncing ball orbit contribution as shown in the inset of Fig. 3.

An interesting result is obtained when we include the contribution of the bouncing ball orbits in the usual spectral unfolding scheme [19] for the statistical analysis. A naive analysis using just the Weyl formula for $N^{\rm sm}(k)$ would show large deviations from the GOE prediction in the usual statistical measures. This is consistent with previous numerical results and analyses [7]. The inclusion of $N^{\rm bb}(k)$ in the unfolding scheme changes the results quantitatively. For the unfolded spectrum the correlation between neighboring level spacings is $C=-0.298\pm0.030$, in agreement with the GOE predicting $C=-0.271$ [19]. For the nearest-neighbor spacing distribution (NND) we performed a best fit with the Brody distribution [20], which is given by $P(s)=c_1 s^{\omega}\times\exp(-c_2 s^{\omega+1})$. The constants c_1 and c_2 are obtained by the normalization constraints and ω is the repulsion parameter. The best fit yields $\omega=0.82\pm0.07$, the result is displayed in the upper part of Fig. 4. For the NND the effect is not very large, therefore, we only display the results for the unfolding where $N^{\rm Weyl}+N^{\rm bb}$ defines the average cumulative level density. But for the $\Delta_3(L)$ statistics the difference is striking. The presence of marginally stable periodic orbits dramatically changes the rigidity of the spectrum for large values of L (measured in terms of the mean level spacing). Proper handling of these orbits—as done here—brings the spectrum back to the expected GOE-like behavior of classically chaotic systems. Moreover, the $\Delta_3(L)$ statistics very closely follows the GOE prediction up to $L=20$, where it saturates, as predicted by Berry [21].

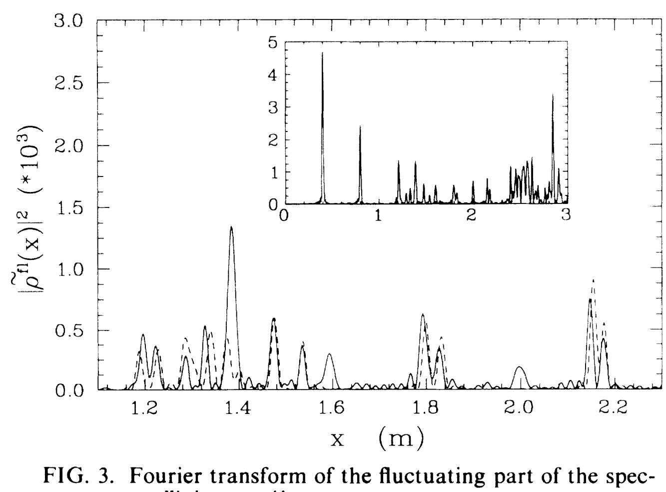

FIG. 3. Fourier transform of the fluctuating part of the spectrum $\rho(k)-\rho^{\rm Weyl}(k)-\rho^{\rm bb}(k)$. The result of a reconstruction of the Fourier power spectrum using the thirty shortest classical periodic orbits and Gutzwiller's trace formula is given by the dashed curve. Inset: The Fourier power spectrum of the measured density $\rho(k)$ minus $\rho^{\rm Weyl}(k)$.

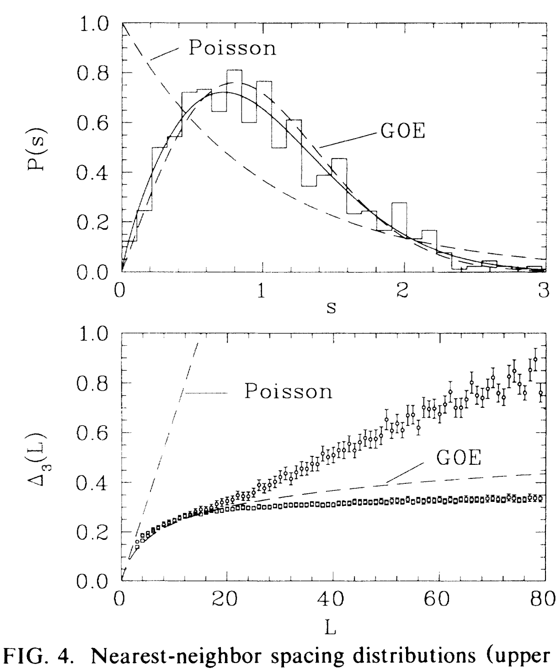

FIG. 4. Nearest-neighbor spacing distributions (upper part). The histogram corresponds to the data and the dashed lines to the theoretical predictions. The solid line shows the best fit with the Brody distribution (see text). Lower part: $\Delta_3(L)$ statistic of the experimental data set compared with theoretical predictions (dashed lines). The circles (squares) derive from the unfolded spectrum with (without) the bouncing ball orbit contribution.

In conclusion, we have presented data comprising the complete spectrum of the first 1060 eigenvalues of the Bunimovich stadium. The data are in good agreement with semiclassics. The apparent discrepancy with the GOE was explained by the presence of nonisolated marginally stable periodic orbits.

We thank H. Alt, O. Bohigas, E. Bogomolny, B. Eckhard, G. Mitchell, K. Müller, T. Rietdorf, U. Smilansky, M. Sieber, and H.-J. Stöckmann for discussions, suggestions, and valuable help. This work is part of the Sonderforschungsbereich 185 "Nichtlineare Dynamik" of the Deutsche Forschungsgemeinschaft.

[1] H.-J. Stöckmann and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990).
[2] See, e.g., O. Bohigas, in *Chaos and Quantum Physics*, edited by M.-J. Giannoni *et al.* (Elsevier, Amsterdam, 1991), pp. 89–199.
[3] S. W. McDonald and A. N. Kaufman, Phys. Rev. Lett. **42**, 1189 (1979); Phys. Rev. A **37**, 3067 (1988).
[4] E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).
[5] O. Bohigas *et al.*, Phys. Rev. Lett. **52**, 1 (1984).
[6] E. B. Bogomolny, Physica (Amsterdam) **31D**, 169 (1988).
[7] A. Shudo and Y. Shimizu, Phys. Rev. A **42**, 6264 (1990).
[8] M. Feingold *et al.*, Phys. Lett. A **146**, 199 (1990).
[9] T. Cheon *et al.*, Phys. Rev. A **44**, R809 (1991).
[10] E. Doron *et al.*, Phys. Rev. Lett. **65**, 3072 (1990).
[11] C. H. Lewenkopf *et al.*, Phys. Rev. A **45**, 2635 (1992).
[12] S. Sridhar, Phys. Rev. Lett. **67**, 785 (1991).
[13] J. Stein and H.-J. Stöckmann, Phys. Rev. Lett. **68**, 2867 (1992).
[14] K. Alrutz-Ziemssen *et al.*, Part. Accel. **29**, 53 (1990).
[15] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).
[16] In order to obtain the smooth part of $\rho(k)$ we fitted a second-order polynomial by the cumulative level density $N(k)=\int_0^k dk'\rho(k')$ [see D. Wintgen, Phys. Rev. Lett. **58**, 1589 (1987)]. This polynomial coincides with the Weyl formula to within 0.18% for the highest measured frequency, which corroborates the assumption that there are no missing levels.
[17] This unpublished result obtained by R. G. Littlejohn was kindly communicated to us by E. Bogomolny and later independently derived by M. Sieber and U. Smilansky.
[18] M. Sieber and F. Steiner, Physica (Amsterdam) **44A**, 248 (1990).
[19] O. Bohigas and M.-J. Giannoni, in *Mathematical and Computational Methods in Nuclear Physics*, Lecture Notes in Physics Vol. 209, edited by J. S. Dehesa *et al.* (Springer, Berlin, 1984), p. 1.
[20] T. A. Brody *et al.*, Rev. Mod. Phys. **53**, 418 (1981).
[21] M. V. Berry, Proc. R. Soc. London A **400**, 229 (1985).