Bose-Einstein Condensation in Relativistic Systems in Curved Space as Symmetry Breaking

D. J. Toms
Physics Department, University of Newcastle upon Tyne, Newcastle upon Tyne NE1 7RU, United Kingdom
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A treatment of Bose-Einstein condensation in a static spacetime with a possible spatial boundary is given. An interpretation in terms of symmetry breaking to give a nonconstant scalar-field vacuum expectation value is provided. Applications to condensation in cavities in flat spacetime, and to curved spacetime, are discussed.

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Bose-Einstein condensation (BEC) has been of continuing interest to physicists since its original discovery in nonrelativistic systems [1,2]. It provides at least a partial understanding for the behavior of liquid helium at low temperatures as suggested originally by London [3]. The phenomenon of BEC in relativistic systems has also been considered (see, for example, Refs. [4–6]). More recently BEC has been examined from the viewpoint of modern quantum field theory at finite temperature and density [7–11]. In particular, it was shown that a proper account of antiparticles must be taken and, unlike the nonrelativistic case, BEC can occur at high temperatures. The other important aspect of the field-theory approach to BEC is that the accumulation of particles in the ground state may be understood as spontaneous symmetry breaking in the sense that the vacuum expectation value of the scalar field becomes nonzero above a critical temperature [8,9].

The work referenced so far considers relativistic systems in flat spacetime with no boundaries present. The case of flat spaces with boundaries was originally motivated by a desire to understand the behavior of liquid helium in thin films (see Refs. [12,13] for example). Nonrelativistic systems in cubical cavities with a variety of boundary conditions have been studied extensively by Pathria and his co-workers. (This work, with original references, is reviewed in [14].) Relativistic BEC in a cavity has also been treated [15]. The generalization from flat to curved space has been considered [16], beginning with nonrelativistic bosons in the static Einstein universe, in which the spatial section is a three-sphere. The generalization to relativistic particles in the Einstein universe was performed in Refs. [17,18], and for higher-dimensional spheres in [19]. Some of these results will be recovered as special cases of the results in the present paper.

Our basic reason for dealing with cubic enclosures or spheres is that the eigenvalues of the Laplacian are known, and it is then possible by a variety of summation techniques to obtain an exact result for the partition function, or to evaluate it approximately in certain limiting cases. For more general spaces, where the eigenvalues of the Laplacian may not be known explicitly, other methods must be found to study BEC. The purpose of the present paper is to study BEC in the fairly general setting of a static spacetime whose spatial section is an arbitrary manifold with boundary. We will obtain a high-temperature expansion of the effective action in this situation and use it to study BEC. The technique we use is based on that of Actor [20] in flat spacetime, and is similar in spirit, although different in detail to that of Refs. [21–23]. The results are in agreement with those of Kirsten [24,25] who used the methods of Refs. [21–23] to obtain the high-temperature expansion of the thermodynamic potential. In addition, Kirsten [24] has looked at BEC, but without the interpretation of symmetry breaking in our paper. As we will discuss, this interpretation leads to results considerably different from previous treatments. Calculational technicalities, and a variety of applications to specific spacetimes, will be presented elsewhere [26].

Consider a complex scalar field $\Phi(x)$ defined on a spacetime $M \simeq R\times\Sigma$, where Σ is a D-dimensional Riemannian manifold, compact, possibly with a boundary. Assume that M is static with line element

$$ds^2 = dt^2 - g_{ij}(\mathbf{x})dx^i dx^j. \tag{1}$$

The Lagrangian will be chosen to be

$$\mathcal{L} = (\partial^\mu\Phi)^\dagger(\partial_\mu\Phi) - m^2\Phi^\dagger\Phi - U_0(x) - U_1(x)\Phi^\dagger\Phi, \tag{2}$$

where $U_0(x)$ and $U_1(x)$ are independent of the scalar field, but may depend on the geometry of the spacetime. [For example, $U_1(x) = \xi R$, where R is the scalar curvature, is a common choice.] The scalar field is free in the sense that it only interacts with the classical gravitational background. As noted in Refs. [8,9], associated with Eq. (2) is a conserved Noether current, as well as a conserved charge Q, corresponding to infinitesimal gauge invariance. The conserved charge is $Q = i\int_\Sigma d\sigma_x(\Phi^\dagger\dot{\Phi} - \dot{\Phi}^\dagger\Phi)$, where $d\sigma_x$ is the volume element on Σ, and $\dot{\Phi} = \partial\Phi/\partial t$.

The grand partition function Z can be expressed as a Hamiltonian path integral, with the conserved charge incorporated using a Lagrange multiplier μ. Finite temperature may be dealt with by performing a Wick rotation to imaginary time, and performing the path integral over all fields periodic in imaginary time with period $\beta = (kT)^{-1}$ (see Refs. [8,9,27]). Instead of the partition function, we may compute the effective action Γ using the background-field method [28]. After some calculation, we find [26]

$$\Gamma=\beta\int_\Sigma d\sigma_x\{\tfrac{1}{2}|\nabla\varphi|^2+\tfrac{1}{2}(m^2+U_1-\mu^2)\varphi^2\}+\Gamma_++\Gamma_-\,, \tag{3}$$

where

$$\Gamma_\pm=\tfrac{1}{2}\ln\det\left[l^2\left(-\Box+m^2+U_1-\mu^2\pm 2\mu\frac{\partial}{\partial t}\right)\right]. \tag{4}$$

The background field $\Phi(x)=\varphi(\mathbf{x})/\sqrt{2}$ is chosen to be real, and to depend only on the coordinates on Σ. l is a constant with dimensions of length chosen to keep the argument of the logarithm in Eq. (4) dimensionless. If we call Γ_+ the contribution from particles, then Γ_- is the contribution from antiparticles. The two contributions are seen to arise naturally in the context of quantum field theory.

In order to calculate Eq. (4) we will adopt Hawking's [29] definition of ζ-function regularization which involves the eigenvalues of the differential operators in (4). Because of (1), $\Box=\partial^2/\partial t^2+\nabla^2$, where ∇^2 is the scalar Laplacian on Σ. If σ_N denotes the eigenvalues of $-\nabla^2+U_1(\mathbf{x})$ on Σ with the fields subject to the appropriate boundary conditions, then the differential operators in Eq. (4) have eigenvalues

$$\lambda_{Nj}^{\pm}=(2\pi j/\beta\pm i\mu)^2+\sigma_N+m^2\,, \tag{5}$$

where $j=0,\pm1,\pm2,\ldots$. Generalized ζ functions are defined by

$$\zeta_\pm(s)=\sum_{j=-\infty}^{\infty}\sum_N(\lambda_{Nj}^{\pm})^{-s}\,, \tag{6}$$

and we define

$$\Gamma_\pm=-\tfrac{1}{2}\zeta'_\pm(0)+\tfrac{1}{2}\zeta_\pm(0)\ln l^2\,. \tag{7}$$

It is easy to see that $\zeta_-(s)=\zeta_+(s)$, so that $\Gamma_+=\Gamma_-$. [This is also obvious from Eq. (4) since $t\to -t$ is a symmetry of the spacetime.] Because σ_N is not explicitly known, except in certain very special cases, it is not possible to evaluate $\zeta_+(s)$ in closed form. However, it is possible to obtain an approximation valid at high temperature by a generalization of Actor's [20] method from flat to curved spacetime.

Define

$$\Theta(t)=\sum_N\exp\left(-\frac{t\beta^2\sigma_N}{4\pi^2}\right), \tag{8}$$

which is related to the integrated heat kernel for $-\nabla^2+U_1$. (The literature on the heat kernel is vast. The first use in quantum field theory was by Schwinger [30] and DeWitt [28].) It can be noted that high temperature corresponds to $\beta\to 0$. The behavior of $\Theta(t)$ is then found to be related to $t\to 0$. As $t\to 0$, $\Theta(t)$ has the asymptotic expansion

$$\Theta(t)\simeq\left(\frac{t\beta^2}{\pi}\right)^{-D/2}\sum_{k=0,1/2,1,\ldots}\left(\frac{t\beta^2}{4\pi^2}\right)^k\theta_k\,, \tag{9}$$

where θ_k are coefficients determined by the intrinsic and extrinsic geometry of Σ, as well as by the boundary conditions. The result in Eq. (9) allows the high-temperature limit of the generalized ζ function to be obtained. We will give the results only for the case $D=3$, corresponding to a four-dimensional spacetime (other cases and full technical details will be found in Ref. [26]). For $D=3$, we find

$$\Gamma_+=\Gamma_-\simeq-\frac{\pi^2}{90}\beta^{-3}\theta_0-\frac{\zeta_R(3)}{4\pi^{3/2}}\beta^{-2}\theta_{1/2}-\tfrac{1}{24}\beta^{-1}[\theta_1+(2\mu^2-m^2)\theta_0]+\cdots \tag{10}$$

if only the dominant terms at high temperature are kept. Terms of order $\ln\beta$ and those which vanish as $\beta\to 0$ may be easily obtained if needed. The flat-spacetime result agrees with Ref. [8]; the curved-spacetime result agrees with Refs. [24,25].

It is important to clarify in what sense we mean that the temperature is high. In flat spacetime, $T\gg m$ is assumed. In curved spacetime, in addition to $T\gg m$, we require $T\gg|R|^{1/2}$, where $|R|$ is the magnitude of a typical curvature of the spacetime. If this last approximation is not made, then a term of zeroth order in T, such as R, could be of the same or of greater order than T^2. The expansion in Eq. (10) would not be consistent in this case. It would appear difficult to remove this restriction for general spacetimes (see Ref. [18] for the special case of the Einstein static universe).

The vacuum, or ground, state is the value of $\varphi(\mathbf{x})$ which minimizes the effective action. From Eq. (3), the stationary point is the solution to

$$-\nabla^2\varphi+(m^2+U_1-\mu^2)\varphi=0\,, \tag{11}$$

subject to the appropriate boundary conditions. Let $\{\varphi_N(\mathbf{x})\}$ be a complete set of solutions to

$$[-\nabla^2+U_1(\mathbf{x})]\varphi(\mathbf{x})=\sigma_N\varphi(\mathbf{x})\,, \tag{12}$$

$$\int_\Sigma d\sigma_x\,\varphi_N(\mathbf{x})\varphi_{N'}(\mathbf{x})=\delta_{NN'}\,. \tag{13}$$

It should be noted that $\mu^2\le\sigma_0+m^2$ is assumed here in order that the effective action does not involve negative eigenvalues. We may expand $\varphi(\mathbf{x})$ in Eq. (11) as

$$\varphi(\mathbf{x})=\sum_N c_N\varphi_N(\mathbf{x})\,, \tag{14}$$

for some expansion coefficients C_N. From Eq. (11), using the linear independence of the $\varphi_N(\mathbf{x})$, it follows that

$$C_N(\sigma_N+m^2-\mu^2)=0\,. \tag{15}$$

If $\mu^2<\sigma_0+m^2$, where σ_0 is the smallest eigenvalue, then the only solution to (15) is for $C_N=0$. This corresponds to $\varphi(\mathbf{x})=0$, which represents the unbroken symmetry phase. If $\mu^2=\sigma_0+m^2$, then C_0 is not determined by (15); however, $C_N=0$ for all $N\neq 0$. Thus, in this case, we can have a nonzero solution $\varphi(\mathbf{x})=C_0\varphi_0(\mathbf{x})$ which represents symmetry breaking.

In order to see how symmetry breaking is possible, consider the expectation value of the charge operator Q which is given in terms of the effective action by

$$Q=-\frac{1}{\beta}\frac{\partial\Gamma}{\partial\mu}. \tag{16}$$

Using the high-temperature expansion (10) in (3) leads to

$$Q=Q_0+Q_1, \tag{17}$$

where

$$Q_0=\mu\int d\sigma_x\varphi^2(\mathbf{x}), \tag{18}$$

$$Q_1\simeq\tfrac{1}{3}\mu VT^2. \tag{19}$$

(We have used the result $\theta_0=V$, where V is the volume of Σ. This holds irrespective of the boundary conditions on the fields [31-34].) If T is high enough, it is always possible to have $\varphi(\mathbf{x})=0$ and satisfy

$$Q\simeq\tfrac{1}{3}\mu VT^2, \tag{20}$$

where $\mu^2<\sigma_0+m^2$. This is the unbroken phase. We can interpret Q_0 as the charge associated with particles in the ground state, and Q_1 as the charge associated with particles in excited states. As T decreases, since the total charge is fixed, μ must increase until we reach the temperature at which

$$\mu^2=\sigma_0+m^2. \tag{21}$$

The temperature at which (21) holds defines the critical temperature T_c. As discussed in the previous paragraph, this allows a nonzero ground state, and by (18) an accumulation of charge in the ground state. From (20) and (21), the critical temperature is given by

$$T_c=(3Q/V)^{1/2}(\sigma_0+m^2)^{-1/4}. \tag{22}$$

For $T\leq T_c$, it is easily seen that

$$Q_0=Q[1-(T/T_c)^2], \tag{23}$$

which is identical to the result in flat spacetime [8]. This result therefore has a universal character, irrespective of the spacetime. (Of course the critical temperature is not simply given by the flat-spacetime result in general.) Using $\varphi(\mathbf{x})=C_0\varphi_0(\mathbf{x})$ in (18), and solving for C_0 using (23) gives the broken symmetric ground state

$$\varphi(\mathbf{x})=(\tfrac{1}{3}V)^{1/2}(T_c^2-T^2)^{1/2}\varphi_0(\mathbf{x}) \tag{24}$$

for $T\leq T_c$. One significant difference with the flat-spacetime result is that the charge density in the ground state will not be constant if $T<T_c$, but will have a spatial variation determined by the behavior of $\varphi_0(\mathbf{x})$.

As the first illustration of the general results, consider a scalar field confined to a rectangular cavity of sides L_1,L_2,L_3 with Dirichlet boundary conditions ($\varphi_N=0$ on the walls of the cavity). With $U_1=0$, it is easy to show that the eigenfunction

$$\varphi_0(\mathbf{x})=\left(\frac{8}{V}\right)^{1/2}\sin\left(\frac{\pi x_1}{L_1}\right)\sin\left(\frac{\pi x_2}{L_2}\right)\sin\left(\frac{\pi x_3}{L_3}\right) \tag{25}$$

has the lowest eigenvalue of

$$\sigma_0=\pi^2\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}+\frac{1}{L_3^2}\right). \tag{26}$$

The expectation value of the scalar field is forbidden to be constant (apart from zero) by the boundary conditions. The critical temperature follows using (26) in (22). A similar result holds for the three-torus where the field satisfies antiperiodic boundary conditions. In either case, because the exact eigenvalues are explicitly known, and are so simple, it is possible to evaluate the generalized ζ function explicitly.

A simple example where it is possible to obtain the eigenvalues and eigenfunctions, but not to calculate the ζ function explicitly is a scalar field confined to the inside of a spherical cavity of radius a with Dirichlet boundary conditions. The exact eigenvalues are $\sigma_{nlm}=(z_{l+1/2\,n})^2/a^2$, where $z_{\nu n}$ denotes the nth positive zero of the Bessel function of order ν. The lowest eigenfunction is

$$\varphi_0(\mathbf{x})=(2\pi a)^{-1/2}r^{-1}\sin(\pi r/a), \tag{27}$$

with $\sigma_0=\pi^2/a^2$ the lowest eigenvalue. If von Neumann boundary conditions are imposed ($\partial\varphi/\partial r=0$ at $r=a$), then $\sigma_0=x_0^2/a^2$, where $x_0\simeq 4.4934$ is the smallest nonzero positive solution to $\tan x=x$.

Finally, we look at the Einstein static universe with $U_1=\xi R=6\xi a^{-2}$, where a is the radius of the three-sphere. The lowest eigenvalue of $-\nabla^2$ is 0, corresponding to a constant eigenfunction. Thus, $\sigma_0=6\xi a^{-2}$. The minimally coupled case ($\xi=0$) therefore leads to a critical temperature identical to that in flat spacetime in agreement with Ref. [18]. The conformally coupled case ($\xi=\frac{1}{6}$) gives a critical temperature of $T_c=(3Q/V)^{1/2}\times(m^2+a^{-2})^{-1/4}$, in agreement with Ref. [15]. In either case, if $T<T_c$, the symmetry breaks to a constant solution given by (24) with $\varphi_0=V^{-1/2}$, as in flat spacetime. Instead of taking Σ to be the three-sphere, if we choose instead the three-sphere with antipodal points identified, then $\Sigma\simeq P_3(R)$ is the projective three-sphere. Imposing periodic boundary conditions at antipodal points leads to a critical temperature identical to that in the Einstein static universe. If, however, an antiperiodic boundary condition is imposed, then $\varphi_0=V^{-1/2}$ is no longer an allowed solution. Parametrizing $P_3(R)$ by angles $0\leq\theta,\chi,\phi\leq\pi$, we have $\varphi_0=2V^{-1/2}\cos\chi$ and $\sigma_0=(6\xi+3)a^{-2}$. Even for the minimally coupled field the result

for the critical temperature changes from its flat-spacetime form, unlike that in the Einstein universe.

The effects of field interactions have not been considered in this paper. In flat Minkowski spacetime they may alter considerably the free-field results [10,11]. For the spacetimes considered in this paper, where the vacuum expectation value of the field is not constant, this means that the effective potential may not be used and the analysis of interactions is much more complicated than for Minkowski spacetime. This problem is currently under investigation. Another point which should be mentioned concerns the application to cosmology. Because we have restricted our attention to static spacetimes, no direct conclusions can be drawn from this paper; however, it should be possible to generalize the method presented here to deal with dynamic spacetimes. Some possible cosmological consequences are discussed in Refs. [10, 11,18].

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