Analytical Derivation of the Scaling Law for the Inverse Participation Ratio in Quasi-One-Dimensional Disordered Systems

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We present a calculation of the inverse participation ratio in finite quasi-one-dimensional samples in the whole range of the scaling parameter within the framework of a one-dimensional nonlinear supermatrix σ model. The results are valid for both thick wires and random band matrices with large bandwidth and so are relevant for quantum chaos problems. The derived form of the scaling law exactly coincides with the empirical expression deduced earlier from results of computer simulations.

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The one-parameter scaling hypothesis put forward in [1] can be formulated as the requirement that all physical properties of the disordered solid of finite size should be expressed only in terms of the ratio between the localization length for the infinite sample, ξ_∞, and the size of the sample L. The remarkable feature of the quasi-one-dimensional disordered systems is that not only do they allow for rather detailed numerical investigations of their scaling properties [2], but also the explicit calculation of scaling functions becomes possible under some conditions. This was recently demonstrated by Zirnbauer [3] who managed to calculate such a function for the average conductance.

Recently the interest in the scaling properties of random quasi-one-dimensional systems was greatly stimulated not only by electron transport theory itself, but also by new developments in the field of "quantum chaos." It was found that for certain deterministic Hamiltonian systems whose classical dynamics is chaotic, quantum effects suppress classical diffusion in the phase space in a way analogous to suppression of the diffusion of a quantum particle by a random potential, known as the Anderson localization phenomenon. The prototype system where this effect of "dynamical localization" was discovered for the first time was the so-called kicked rotator (KR) (see review [4]), for which a formal connection with the Anderson tight-binding model has been found [5]; later the effect was shown to occur in a number of other systems, e.g., the hydrogen atom in a monochromatic field [6].

In an appropriate basis, the matrix of the "evolution operator" **U** for the KR that relates values of the wave function in one period of perturbation appears to have a band structure with pseudorandom elements within the band [4]. The width of the band proves to be large in the chaotic regime. These observations lead to the conjecture that statistical properties of eigenvectors and eigenvalues of the KR model can also be typical for the ensemble of random band matrices (RBM) introduced in [7].

Intensive numerical simulations [8,9] gave evidence for this conjecture and revealed some nice scaling relations that hold for both the KR and RBM. A set of generalized localization lengths ξ_q, $q=1,2,\ldots$, was introduced according to the definition

$$\ln\xi_q = \frac{1}{1-q}\ln\sum_k \overline{|\Psi_k(n)|^{2q}\delta(E-E_k)}\,. \qquad (1)$$

Here E_k is an energy level of the "Hamiltonian" (band matrix), $\Psi_k(n)$ stands for the component of the corresponding eigenfunction at given site $n=1,2,\ldots,N$, N being the sample length (matrix size), and the overbar denotes an averaging both over the disorder and over all sites of the sample. Such a quantity ξ_q^{-1} for $q=2$ coincides with the frequently used "inverse participation ratio" (IPR) which is related to the probability for a quantum particle to stay at a given site for infinite time.

It was found that in a wide range of parameters numerical results are well described by the following empirical scaling law [8]:

$$\frac{\beta_q}{1-\beta_q} = Cx^*\,, \qquad (2)$$

where $\beta_q = \xi_q/\xi_q^{\rm ref}$ and $x^* = b^2/N$, b being the bandwidth of the matrix, C being some model-dependent constant, and $\xi_q^{\rm ref}$ being the generalized localization length for some reference ensemble taken to be a Gaussian orthogonal ensemble for which $\xi_q^{\rm ref}$ is known explicitly (e.g., $\xi_2^{\rm ref}=N/3$). In a very recent paper [10] a relation equivalent to Eq. (2) was claimed to be true also for genuine one-dimensional tight-binding models—those of Anderson and Lloyd.

In contrast to the numerous computer investigations of the KR and RBM, analytical results are much less abundant, and so the status and the range of validity of conjectures deduced from numerical simulations are not clear. As to the KR problem, we should mention that some analysis becomes possible on the basis of the quasiclassical approximation [11]. However, the region of parameters where quantum effects prevail has so far been beyond the reach of this approach.

The theoretical consideration of RBM was started in [12], where the investigation of statistical properties of RBM was reduced to the analysis of a supermatrix (graded) nonlinear σ model. The same σ model was derived earlier in the course of the investigation of electron localization in thick wires [13].

In the present paper we give, for the first time, an analytical derivation of the scaling law equation (2) for the inverse participation ratio ξ_2^{-1} by using the graded nonlinear σ-model formulation. Though our results are equally applicable to both RBM and thick wires, we mostly keep with the RBM terminology.

We consider the ensemble of random bandlike $N\times N$ ($N\gg 1$) real symmetric matrices $\hat{H}$ whose elements $H_{ij}=H_{ji}$ are distributed independently around zero according to the Gaussian law, with the variances $\langle H_{ij}^2\rangle_{\rm av}=\frac{1}{2}A_{ij}(1+\delta_{ij})$ depending on the distance $r=|i-j|$. When $r>b$ the function $A_{ij}=a(r)$ is assumed to be (exponentially) fast decreasing, b playing the role of the effective bandwidth. We also adopt the normalization condition $B_0=\sum_{-\infty}^{\infty}a(r)\propto 1$ at $b\gg 1$.

Our starting point is the following well-known expression for the "position-dependent" IPR $P(n)$ in terms of the two-point Green function [14]:

$$P(n)=\frac{1}{\pi\rho}\lim_{\epsilon\to 0}\epsilon\langle K(n,n)\rangle_{\rm av}\,,$$
$$K(l,m)=\langle l|\frac{1}{E+i\epsilon-\hat{H}}|l\rangle\langle m|\frac{1}{E-i\epsilon-\hat{H}}|m\rangle\,, \tag{3}$$

where the brackets $\langle\cdots\rangle_{\rm av}$ denote the disorder averaging, and ρ is the density of states.

By using the supersymmetry approach [13,15] it is possible to perform the averaging in Eq. (3) and, provided the parameter b is large, to reduce the quantity $\langle K(l,m)\rangle_{\rm av}$ to the form [12,16]

$$\langle K(l,m)\rangle_{\rm av}=\left(\frac{\pi\rho}{4}\right)^2\int\prod_{i=1}^{N}d\mu(\hat{Q}_i)F(\hat{Q}_l,\hat{Q}_m)\exp[-S\{\hat{Q}\}]\,, \tag{4}$$

$$F(\hat{Q}_l,\hat{Q}_m)={\rm Str}[\hat{k}\hat{Q}_l^{11}]{\rm Str}[\hat{k}\hat{Q}_m^{22}]+2\delta_{lm}{\rm Str}[\hat{k}\hat{Q}_l^{12}\hat{k}\hat{Q}_l^{21}]\,, \tag{5}$$

where

$$S\{\hat{Q}\}=\frac{\gamma}{2}\sum_i{\rm Str}(\hat{Q}_i-\hat{Q}_{i+1})^2+i\tilde{\epsilon}{\rm Str}\sum_i\hat{Q}_i\hat{L}\,,$$
$$\gamma=\frac{(\pi\rho)^2}{8}\sum_{r=-\infty}^{\infty}a(r)r^2\propto b^2\,,$$
$$\tilde{\epsilon}=\frac{\pi\rho\epsilon}{2},\quad \pi\rho=\frac{1}{B_0}(2B_0-E^2)^{1/2}\,, \tag{6}$$
$$\hat{Q}=\begin{pmatrix}\hat{Q}^{11} & \hat{Q}^{12}\\ \hat{Q}^{21} & \hat{Q}^{22}\end{pmatrix},\quad \hat{L}=\begin{pmatrix}\hat{I}_4 & 0\\ 0 & -\hat{I}_4\end{pmatrix},\quad \hat{k}=\begin{pmatrix}\hat{I}_2 & 0\\ 0 & -\hat{I}_2\end{pmatrix}.$$

Here the symbol Str stands for the supertrace [13],

$$I_n={\rm diag}(\underbrace{1,1,\ldots,1}_{n})$$

and the integration in Eq. (4) goes over supermatrices $\hat{Q}_i=-i\hat{T}_i^{-1}\hat{L}\hat{T}_i$, with matrices $\hat{T}_i$ satisfying the condition $\hat{T}_i^{\dagger}\hat{L}\hat{T}_i=\hat{L}$ and forming the graded Lie group UOSP(2,2/2,2) [see the review [15] for their properties, including the invariant measure $d\mu(\hat{Q})$].

The action equation (6) defines the one-dimensional nonlinear graded σ model. Its continuous version was investigated for the first time in the context of electron localization in thick infinitely long wires [11], with the parameter γ being proportional to the bare diffusion constant and $\tilde{\epsilon}$ being related to the external frequency.

In the present context we find it to be more convenient to investigate the discrete version of the model equation (4) going to the continuous limit at the very end. We should mention that a very detailed study of such a model was performed on the Bethe lattice [4] in the limit of infinite sample. Since the one-dimensional lattice can be considered as the limiting case of the Bethe lattice, we can make use of the intermediate expressions derived in these profound papers.

The procedure suggested in [14] can be briefly outlined as follows. The one-dimensional structure of the integrand in Eq. (4) allows one to write the following expression for the position-dependent IPR:

$$P(n)=\frac{\pi\rho}{16}\lim_{\epsilon\to 0}\epsilon\int d\mu(\hat{Q})Y(\hat{Q};N-n)F(\hat{Q},\hat{Q})Y(\hat{Q};n)\exp[-i\tilde{\epsilon}{\rm Str}(\hat{Q}\hat{L})]\,, \tag{7}$$

where the function $Y(\hat{Q};n)$ satisfies the recurrence equation

$$Y(\hat{Q}';n+1)=\int d\mu(\hat{Q})Y(\hat{Q};n)L(\hat{Q},\hat{Q}'),\quad Y(\hat{Q};0)=1\,,$$
$$L(\hat{Q},\hat{Q}')=\exp\{-\tfrac{1}{2}\gamma{\rm Str}(\hat{Q}-\hat{Q}')^2-i\tilde{\epsilon}{\rm Str}(\hat{Q}\hat{L})\}\,. \tag{8}$$

By using the Efetov parametrization [13,14] of the matrices $\hat{Q}$ it is possible to show that $Y(\hat{Q};n)$ depends only on the "eigenvalues" $\lambda_{1,2},\lambda$ of the block $\hat{Q}^{12}$ and to perform the integration over the remaining degrees of freedom. The crucial simplification occurs when $\epsilon\to 0$ since the only nonvanishing contribution to the integral over $\lambda_{1,2}$ comes from the regions where either $\lambda_1\sim\epsilon^{-1}\gg\lambda_2,\lambda$ or $\lambda_2\propto\epsilon^{-1}\gg\lambda_1,\lambda$. In this asymptotic domain the function $Y(\lambda_1,\lambda_2,\lambda;n)$ proves to be dependent only on the variable $z=\lambda_{1,2}\epsilon$ and performing the computation one finds

$$P(n)=3\pi\rho\int_0^{\infty}dz\,Y(z;n)Y(z;N-n)\exp[-\pi\rho z]\,, \tag{9}$$

$$Y(z;n+1)=\int_0^{\infty}dz_1L_\gamma(z_1)Y(zz_1;n)\exp[-\pi\rho zz_1],$$
$$Y(z;0)=1\,. \tag{10}$$

The particular form of the kernel $L_\gamma(z_1)$ can be found in [14], but it is actually unimportant for our purpose. The relevant properties of this kernel are

$$\int_0^\infty dz_1 L_\gamma(z_1) = 1 ,$$
$$\lim_{\gamma\to\infty} L_\gamma(z_1) = \delta(z_1 - 1) , \quad (11)$$
$$\int_0^\infty L_\gamma(z_1)(z_1-1)^2 dz_1|_{\gamma\gg 1} = (2\gamma)^{-1} .$$

Since $\gamma \propto b^2 \gg 1$ we can replace the integral recurrence relation equation (10) by the differential one. Moreover, it follows from general physical arguments that the IPR must be of the order of $P \sim z_m = \max(\xi_\infty^{-1}, N^{-1})$. Taking into account that $\xi_\infty \propto b^2 \gg 1$ [12] we see that $z_m \ll 1$. From this we conclude that the integration over z in Eq. (9) is effectively cut off by the fast decrease of the functions $Y(z;n)$ for $z > z_m$. Therefore, for our purpose it is sufficient to know the function $Y(z;n)$ in the domain $z \ll 1$.

At this point is is convenient to pass to the continuous limit, that is, to introduce the continuous variable $\tau = n/4\gamma$, $0 \le \tau \le x \equiv N/4\gamma$, instead of the discrete variable $0 \le n \le N$ and to substitute $(1/4\gamma)\partial Y(z,\tau)/\partial\tau$ for $Y(z;n+1) - Y(z;n)$. Besides, we introduce the new "scaling" variable $y = 4\gamma\pi p z$ and average the position-dependent inverse participation ratio over different sites: $\xi^{-1} = (1/N)\sum_n P(n)$. Remembering $\gamma \gg 1$, $\xi_{\rm GOE}^{-1} = 3/N$, we arrive at the following expressions:

$$\xi^{-1}(x) = \frac{3}{N}\int_0^\infty dy \int_0^x d\tau\, Y(y,\tau) Y(y, x-\tau) \overset{\rm def}{=} \xi_{\rm GOE}^{-1}\beta_2^{-1}(x) , \quad (12)$$

$$\frac{\partial Y(y,\tau)}{\partial\tau} = \hat{R}[Y(y,\tau)], \quad \hat{R} = -y + y^2\frac{d^2}{dy^2}, \quad Y(y,\tau=0) = 1 . \quad (13)$$

The standard way to solve an equation like Eq. (13) is to look for its solution in the form of a generalized Fourier expansion in terms of eigenfunctions of the operator $\hat{R}$. The set of such eigenfunctions that decay at $y \to \infty$ is $f_r(y) = 2y^{1/2}K_r(2y^{1/2})$, $K_r(t)$ being the Macdonald functions, and the corresponding eigenvalues are $\lambda_r = (r^2-1)/4$. The functions $f_r(y)$ with purely imaginary indices $r = i\nu$, $\nu \in [0,\infty)$, form the complete orthonormal set and can be used as a suitable basis for the expansion. Such an expansion is known as the Lebedev-Kontorovich transformation and precise conditions are known under which this transformation exists and is invertible [17]. Making use of the corresponding formulas we find that the solution to Eq. (13) satisfying the initial condition $Y(y,0) = 1$ is given by

$$Y(y,\tau) = 2y^{1/2}\left[K_1(2y^{1/2}) + \int_0^\infty d\nu\, b(\nu) K_{i\nu}(2y^{1/2}) \exp\left\{-\frac{1+\nu^2}{4}\tau\right\}\right] ,$$
$$b(\nu) = \frac{2}{\pi^2}\nu\sinh\pi\nu \int_0^\infty K_{i\nu}(x)[x^{-1} - K_1(x)]\frac{dx}{x} = \frac{2}{\pi}\frac{\nu}{1+\nu^2}\sinh(\pi\nu/2) . \quad (14)$$

When calculating the coefficients $b(\nu)$ the use of the identity

$$x^{-1} - K_1(x) = \int_0^\infty du\, e^{-u}\sin(x\sinh u) \quad (15)$$

has been made together with the formulas of [18(a), 18(b)].

In order to proceed further we notice that instead of the scaling function $\beta_2^{-1}(x)$ defined in Eq. (12) it is more convenient to study its Laplace transform $\beta_L(p) = \int_0^\infty e^{-px}\beta_2^{-1}(x)dx$. Introducing the notations $t = 2y^{1/2}$, $\mu^2 = 4p+1$ we get

$$\beta_L(p) = \frac{1}{2}\int_0^\infty dt\, t Y_L^2(t,p) ,$$
$$Y_L(t,p) = \frac{1}{p}\int_0^\infty du\, u \frac{1+\mu+u^2(\mu-1)}{(u^2+1)^2} J_{\mu-1}(tu) , \quad (16)$$

where $J_\mu(a)$ stands for the Bessel function. The expression for $Y_L(t,p)$ can be obtained after straightforward but lengthy transformations from Eq. (14) by using the formulas of [18(c)–18(f)].

Substituting $Y_L(t,p)$ into the formula for $\beta_L(p)$, changing the order of integration, and using the virtual orthogonality of the Bessel functions,

$$\int_0^\infty dt\, t J_\mu(tu) J_\mu(tu') = \frac{1}{u}\delta(u-u') , \quad (17)$$

we easily find the function $\beta_L(p)$ and immediately restore $\beta_2^{-1}(x)$:

$$\beta_L(p) = \frac{1}{p} + \frac{1}{3p^2} \Rightarrow \beta_2^{-1}(x) = 1 + \tfrac{1}{3}x . \quad (18)$$

Remembering the definition $x = N/4\gamma = (b^2/4\gamma)/x^*$ we see that Eq. (18) takes the form of Eq. (2) with the constant $C = 12\gamma/b^2$. On the other hand, substituting the definition $\beta_2(x) = \xi_2(x)/\xi_2^{\rm ef}$ into Eq. (18) we can rewrite it in the following form

$$\frac{1}{\xi_2(N,b^2)} = \frac{1}{\xi_2(\infty,b^2)} + \frac{1}{\xi_2(N,\infty)} \quad (19)$$

equivalent to the "model-independent" form of the scaling law proposed in [10].

In conclusion we would like to note that the calculation performed in the paper can be easily repeated for a system with broken time-reversal symmetry, that is, for Hermitian RBM ($H_{ij} = H_{ji}^*$) or, in another words, for wires with magnetic field. The result is that the scaling law equation (2) retains its validity, but the constant C becomes twice as small in comparison with the symmetric case. This fact is directly related to the known effect of doubling the localization length of a disordered sample in a magnetic field [13,19].

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