Spontaneous Transition from Flat to Spherical Solitons

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This Letter reports on numerical simulations showing exact, flat soliton solutions to a physical equation (that of Zakharov and Kuznetsov) breaking up into spherical solitons. Depending on the size of the box in which calculations are performed, the breakup can be either direct or via an intermediate, cylindrical soliton stage.

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Solitons are of great interest nowadays. They are found in many different branches of physics. Solitons often have surprising stability properties, such as emerging unscathed from collisions. Solitons having flat, cylindrical, or spherical symmetries are well known. One line of investigation, having a huge literature by now, is that of single- and multiple-soliton behavior within a given class. Thus, for example, much is known about flat-flat soliton collisions. A different problem, less amenable to theoretical analysis, is that of the creation of solitons of one symmetry from those of another. An example of this is the transition from flat to cylindrical solitons.

In this paper we will address the extended problem of the breakup of flat solitons to produce cylindrical and/or spherical solitons, as well as of cylindrical solitons to produce spherical ones. Whereas the initial stages of breakup can sometimes be described analytically, the entire transition from one kind of structure to another cannot (as far as we can see). Here we report some results of numerical investigations, based on a simple model equation often used to describe the dynamics of a strongly magnetized, two-component plasma (completely ionized gas). We find that flat solitons can indeed break up, either directly into spherical solitons or else first into cylindrical and then into spherical solitons.

A change of structure in a plasma medium has of course been considered before, largely under the heading of collapse. Here, however, we will concentrate on transitions from one ordered structure (a one- or two-dimensional soliton) to another (a two- or three-dimensional soliton).

In a three-dimensional, unmagnetized plasma, small-amplitude and flat ion acoustic solitons are stable [1-3]. Their existence has been confirmed experimentally [4]. However, a strong external magnetic field will destabilize these entities. A good model for studying this destabilization and its consequences is furnished by the equation formulated by Zakharov and Kuznetsov [5], which was investigated in some detail in Ref. [6].

The Zakharov-Kuznetsov (ZK) equation for ion acoustic waves and solitons propagating along a very strong external and uniform magnetic field is, for a two-component plasma [6],

$$n_t + nn_x + (\Delta n)_x = 0\,, \tag{1}$$

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2\,. \tag{2}$$

Here n is the normalized deviation of the ion density from the average. Exact solitonlike solutions exist in one, two, and three space dimensions. They depend on the independent variables through the combinations

$$x-ct,\quad \rho=[(x-ct)^2+y^2]^{1/2},$$

$$r=[(x-ct)^2+y^2+z^2]^{1/2},$$

respectively, where

$$\Delta n - (c-n/2)n = 0\,, \tag{3}$$

and Δ is ∂_x^2, $\rho^{-1}\partial_\rho \rho \partial_\rho$, and $r^{-2}\partial_r r^2 \partial_r$ for the three cases.

Equation (3) is an ordinary differential equation and is easily solved numerically to give soliton solutions with n decreasing monotonically away from the center. Only for the one-dimensional case can the soliton solution be written out explicitly:

$$n = 3c\,\mathrm{sech}^2[c^{1/2}(x-ct-x_0)/2]\,. \tag{4}$$

The flat soliton (4) is unstable with respect to nonaligned perturbations. This is shown in Refs. [6,7]. The cylindrical soliton is in turn unstable with respect to axial perturbations [8]. As suggested by this logical progression, the spherical soliton, having no fourth space dimension to be unstable in, is in fact stable. A limited mathematical indication of this last statement is seen by looking at dilations that conserve momentum

$$p = \pi \int n^2 r^2 dr$$

(in this model, normalized n and v are equal and the integrand could be written in the more familiar form nv). Now take the following scaling, consistent with momentum conservation:

$$n \to \lambda^{-3/2} n,\quad r \to \lambda r\,. \tag{5}$$

The energy conserved by (1) is now, for the spherical solitonlike solutions,

$$E(\lambda r, \lambda^{-3/2} n) = I_1/3\lambda^3 - I_2/\lambda^4\,, \tag{6}$$

$$I_1 = \pi \int n^3 r^2 dr,\quad I_2 = \pi \int n_r^2 r^2 dr\,. \tag{7}$$

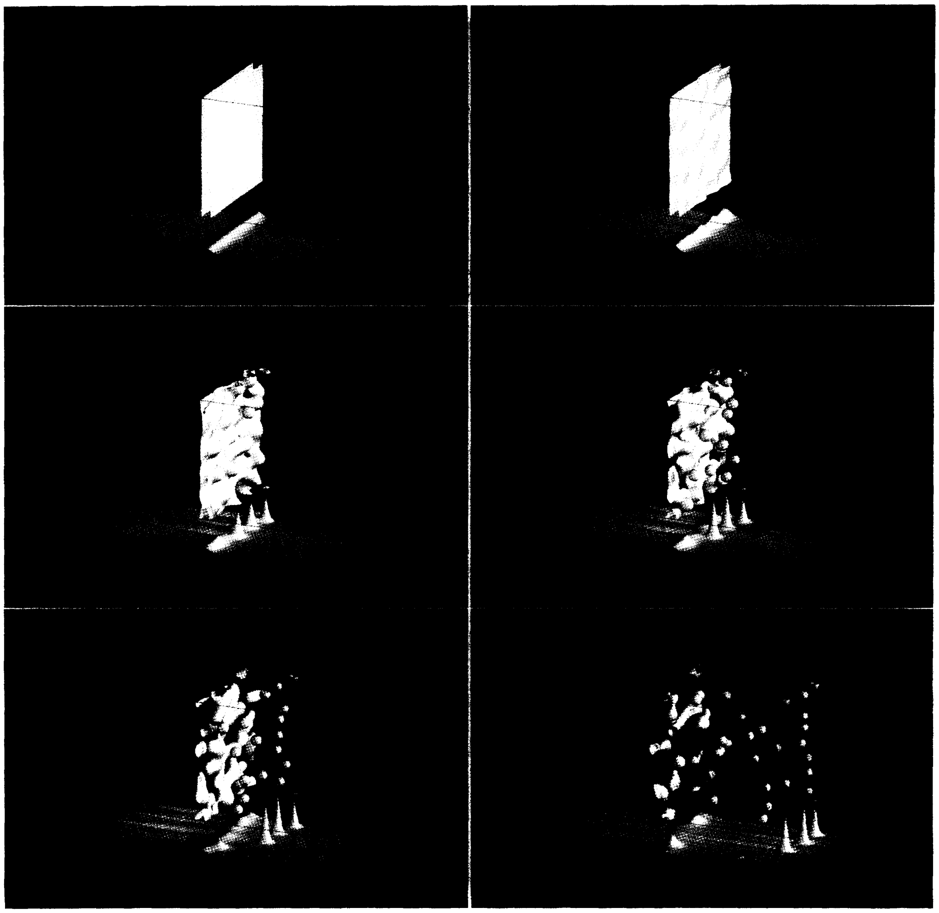

FIG. 1. Consecutive stages in the breakup of a flat soliton when simulations are confined to a box of two equal sides. The coordinate system is moving with the velocity c of the initial soliton. Further details are explained in the text.

Thus

$$\partial E/\partial\lambda = -I_1/\lambda^4 + 4I_2/\lambda^5 . \tag{8}$$

This will have a minimum for the real soliton at $\lambda=1$ if $I_1/I_2=4$. Multiplication of (3) by $r^2 n$ and integration over all space, followed by the same procedure using $r^3 n_r$ and then subtraction of the second resulting equation from the first times $\frac{3}{2}$, does give $I_1=4I_2$.

Spherical solitons are in fact stable with respect to all small-amplitude perturbations, not just dilations [E. Infeld and G. Rowlands (unpublished)]. This will be confirmed numerically in what follows.

The picture indicated by the above considerations is of a flat soliton in three dimensions breaking up either into an array of spherical solitons directly or else into a cylindrical array that will then in turn break up into a spherical array. This is what we expect in three dimensions.

In the two-dimensional picture, flat solitons break up into a stable cylindrical array and this was demonstrated a little while ago [9], when questions about the three-dimensional dynamics were posed. Our numerical results were later repeated and confirmed by others [10].

In our first simulation, Fig. 1, a flat soliton is perturbed by background noise and both perpendicular directions are treated on an equal footing (for the moment just ignore the traces at the bottom of each frame). The value of c in (4) is taken to be 0.5, such that $n_{\max}=1.5$. The gray surfaces are surfaces of $n=0.8$, just two parallel planes for the initial condition expressed by (4) with $t=0$. The noise is of order 10^{-4}. The two sides of the box are long enough to include maximum-growth-rate perturbations as described in Ref. [6], Chap. 8. Not surprisingly, the flat soliton breaks up into a spherical array. The spherical solitons created at later times lag behind the "senior" ones. Shapes and velocities fit spherical solutions of (3). The amplitudes of the resulting solitons are ~ 10. The spherical solitons have been monitored for long times and are indeed stable for these times.

Figure 2 shows the results of a simulation in which the box enclosing the system has uneven sides perpendicular to the motion of the flat soliton. Here the height has been shortened. Maximum-growth-rate perturbations are thus eliminated and the instability grows more rapidly in the y than in the z direction. Cylindrical solitons are found. These gradually break up into spheres.

The logical thing to do now is to bring the height down

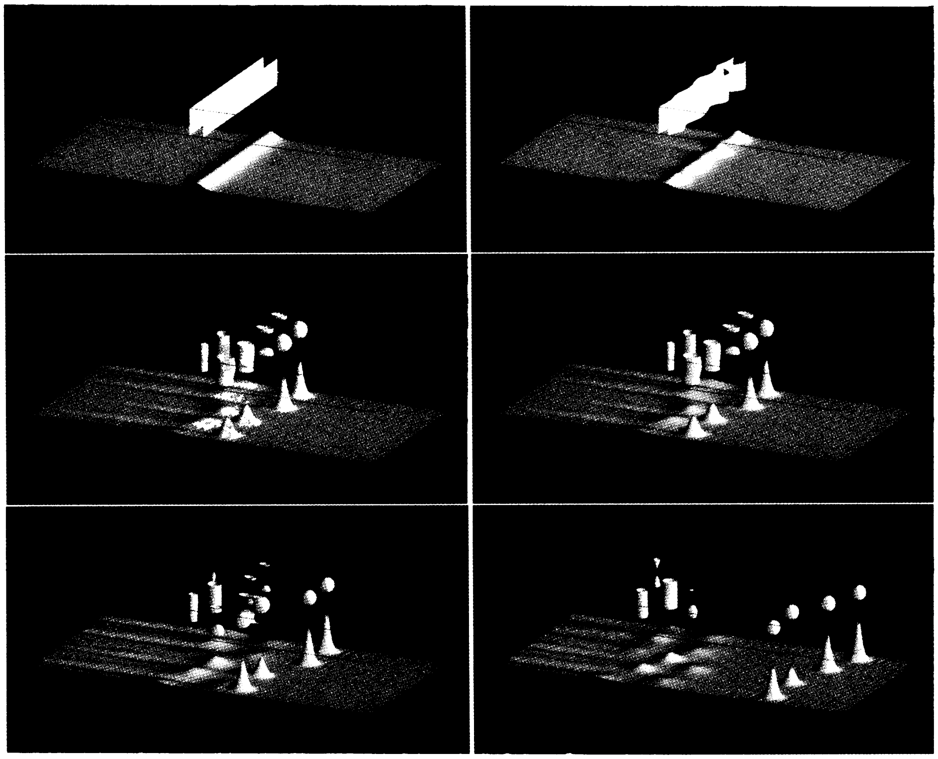

FIG. 2. As in Fig. 1, but with the height of the box considerably shortened.

further in the hope of stabilizing the newly found cylindrical solitons completely. This was the idea behind Fig. 3 in which the threshold is below that for instability of a flat soliton. Indeed the cylindrical solitons last out the simulation. However, when the time was prolonged several times, the cylindrical solitons did break up into spherical arrays. This would suggest that the critical wavelength for unstable perturbations against cylindrical solitons is *much* smaller than for flat solitons, a result we did not anticipate. To date we do not know the formula for the former critical wavelength. However, we can conclude that the cylinders would have to be *very* flat for stability. They would really be disks rather than cylinders.

The illustrations at the bottom of each frame in the figures represent density as a function of x and y for slices through the upper part of the box, very near the top. In this representation both cylinders and spheres appear as "icicles." We see that densities inside spherical solitons tend to be much higher than inside the other two kinds of solitons.

All in all, there seem to be two main possible scenarios for flat solitons in three dimensions: breakup according to the rule

flat soliton → spherical solitons

or

flat soliton → cylindrical solitons,

each cylindrical soliton → spherical solitons.

The third scenario we expected,

flat soliton → cylindrical solitons (stable),

seems doubtful.

This paper completely answers the questions posed by two of the present authors in Ref. [9] about the breakup of flat solitons. The behavior described here is generic and other solitons have been looked at [11].

In numerical simulations we assume periodic boundary conditions in all three dimensions but we keep the boundaries remote enough so that they do not affect the solution significantly. We use the Fourier-Galerkin spectral method for spatial discretization [12]; alternatively, the initial condition is transformed to Fourier space and temporal evolution is implemented in terms of the expansion coefficients. We advance in time the Fourier coefficients of the initial condition. Our code is dealiased because we have found that an aliasing instability develops in simulations with a stationary structure (soliton in moving coordinate system). The time discretization scheme uses the implicit midpoint rule [13],

$$u^{n+1}=u^n+\tau F((u^n+u^{n+1})/2),$$

where $F(u)$ is the right-hand side of the differential equation written in the form $u_t=F(u)$ and τ is the time step. The nonlinear term of the ZK equation is treated iteratively with preconditioning given by an explicit rule.

The accuracy of the results may be measured by the maximum relative errors in the three conserved quanti-

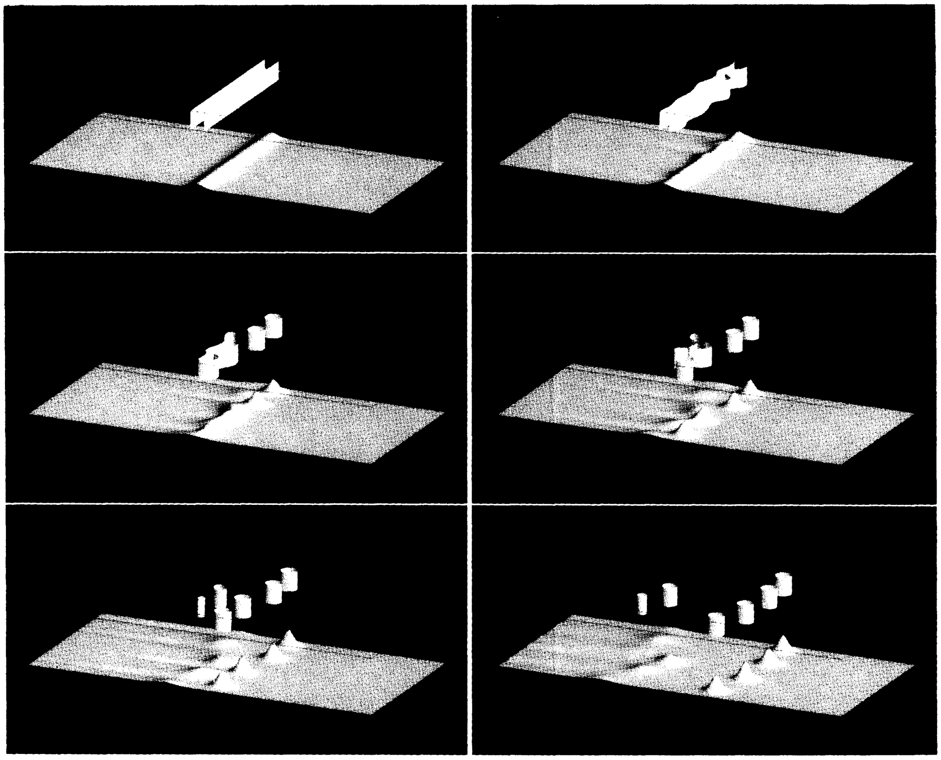

FIG. 3. As above, but with the height of the box further shortened.

ties: mass $\int n\, d^3x$, the x component of the momentum $\int n^2 d^3x$, and the energy $\int(\frac{1}{3}n^3 - n_x^2 - n_y^2 - n_z^2)d^3x$. The errors are $< 10^{-7}$, 6×10^{-5}, and 3×10^{-4}, respectively.

The analysis of the results and the figures were prepared using AVS (Application Visualization System) on the Stardent 3040 computer. The simulations were performed on the same computer using a hand-coded parallelized fast Fourier transform routine. The CPU time of a run varied from 20 h (cubic box) to 100 h (flat box). (This time can probably be decreased significantly.)

The above considerations are distinct from the problem of the collapse of isolated wave packets. For Langmuir turbulent collapse, see Refs. [14–16]. The last reference treats the case of collapse into pancake-shaped objects. For this special case see also Ref. [17]. For recent developments on Langmuir and other types of collapse, see Refs. [17,18] in particular the last paper.

Here we have considered the transition from one ordered structure (a one-dimensional soliton) to another ordered structure (two- or three-dimensional structure).

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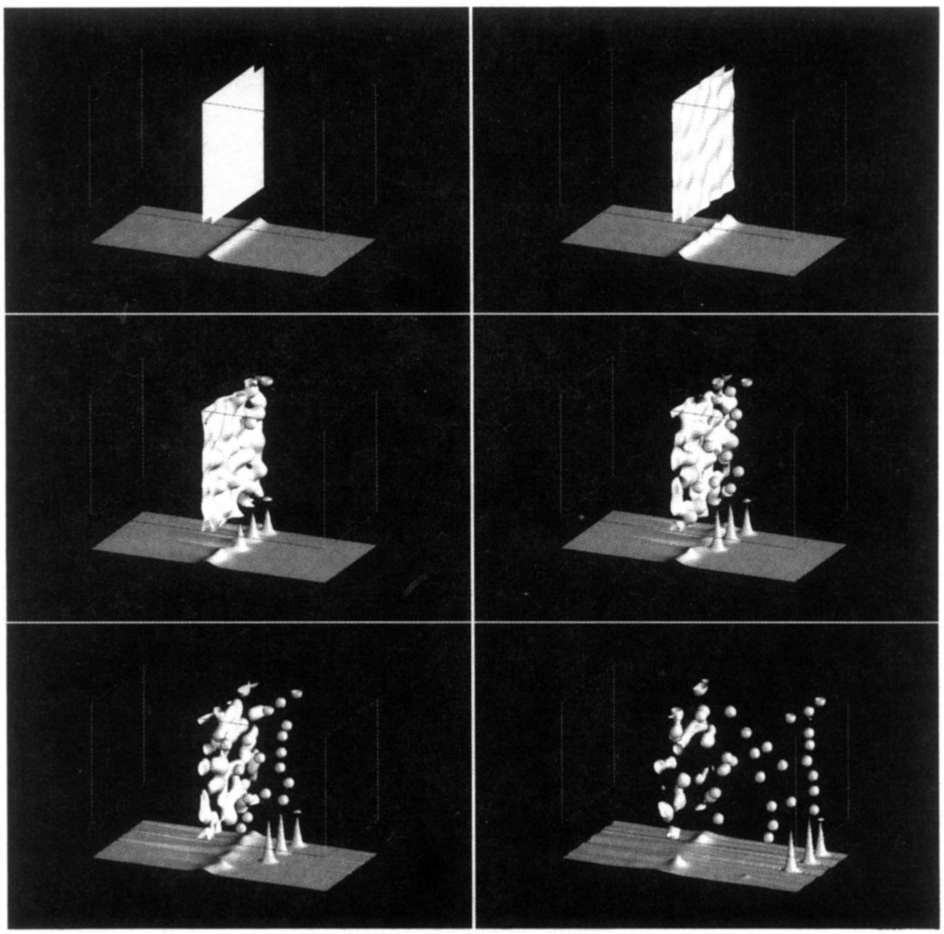

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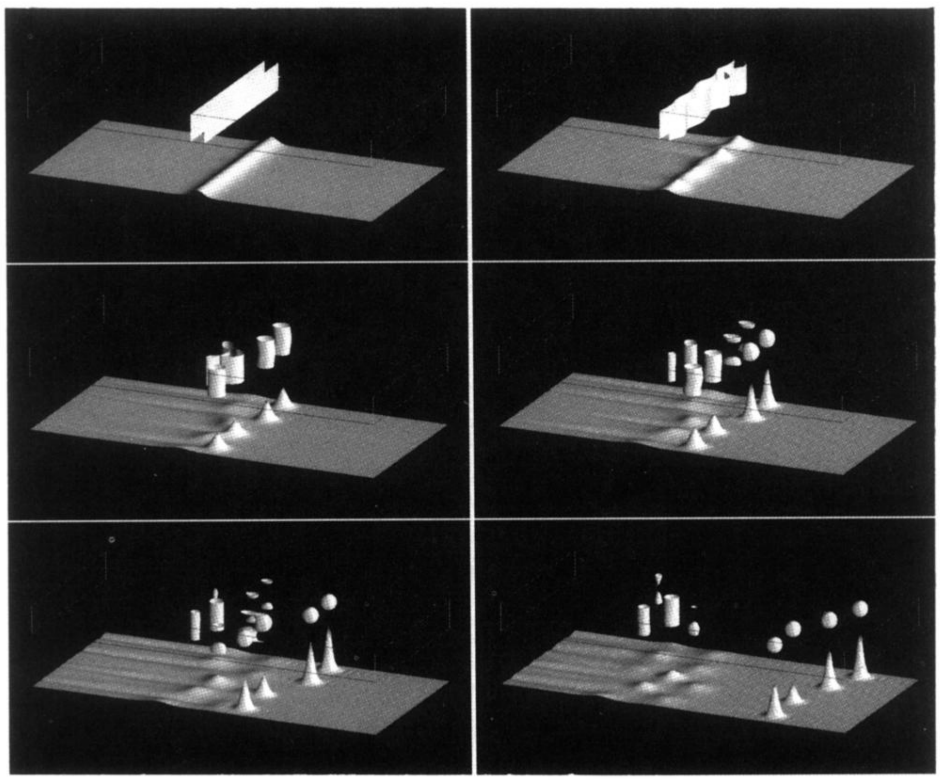

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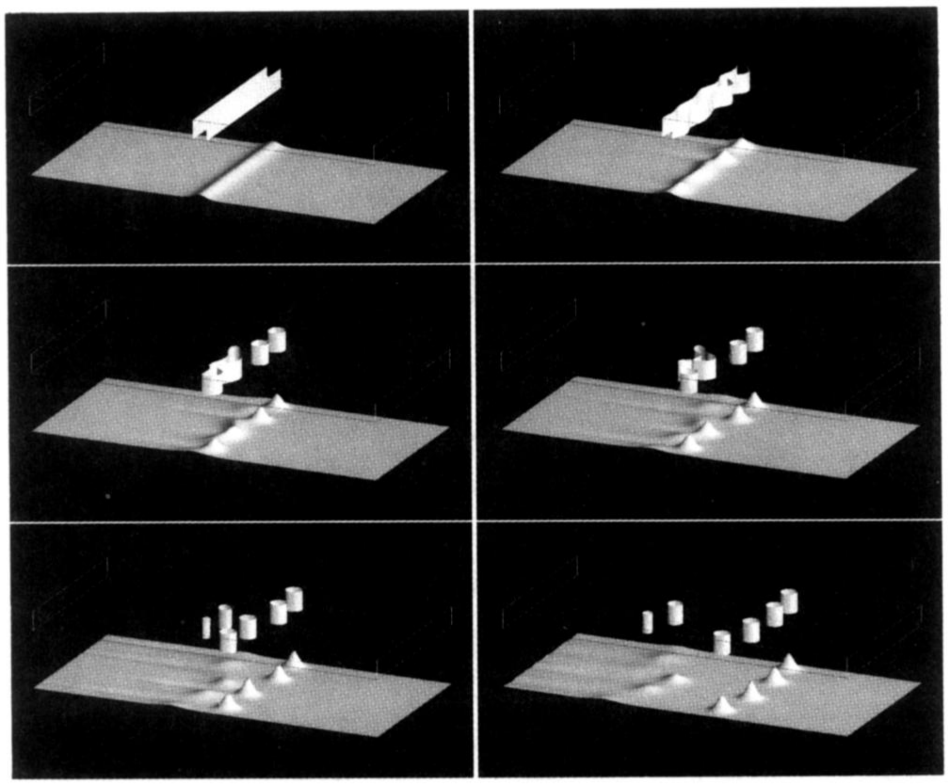

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