# Theory of Anomalous Transport in High-Aspect-Ratio Toroidal Helical Plasmas

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A theoretical model of the anomalous transport in torsatron-heliotron plasmas is developed, based on the current-diffusive interchange instability which is destabilized due to the averaged magnetic hill near the edge. An analytic formula of the transport coefficient is derived. This model explains the large edge transport, the power degradation and energy confinement scaling law, and the enhanced heat-pulse thermal conduction.

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Much work has recently been done on the plasma transport across the magnetic surface in toroidal plasmas. Studies were made on plasmas in stellarators with the magnetic shear and hill, such as torsatron-heliotron devices [1-3]. These plasmas, in which the pressure gradient plays the dominant role in exciting instabilities, may yield knowledge on anomalous transport complementary to tokamaks. Experiments have shown that (I) the energy confinement time $\tau_E$ degrades with power [4-7]. By a comparison study on plasmas of this kind, a scaling law for $\tau_E$ has been proposed [5]. Detailed studies have shown that (II) the effective thermal conductivity $\chi_{\rm eff}$ increases with temperature for a given minor radius $r$ [8]; (III) the profile of $\chi_{\rm eff}$ for a given heating power, however, is an increasing function of $r$ [7,8], and a simple form like $\chi_{\rm eff}\propto T_e^{1.5}/B^2$ is not valid. It is also known that (IV) the thermal transport coefficient that is determined by the heat-pulse propagation, $\chi_{\rm HP}$, is larger than $\chi_{\rm eff}$ [9]. These results have similarities and differences compared to those for tokamaks, and the explanation of these will provide a key to understanding the anomalous transport in toroidal plasmas.

The interchange mode [10] has been thought to be a candidate to explain the anomalous transport in torsatron-heliotron plasmas. This mode can be destabilized, for instance, in the presence of finite resistivity $\eta$. Much effort has been directed to studying the anomalous transport driven by the resistive interchange mode [11-14]. The theoretical methods are the mixing-length model [15], scale-invariance method [16], and one/two-point renormalization technique [17]. The different methods have given the same result on $\chi_{\rm eff}$ from the physics point of view (the differences appear only in a numerical constant) [18]. In spite of these intensive studies, the anomalous transport, characterized by (I)-(IV), remains unexplained.

We have recently investigated the effects of transport coefficients on the interchange instability, such as thermal diffusivity $\chi$, viscosity $\nu$, and especially, the current diffusivity $\lambda$ [19]. (The current diffusivity comes from the electron viscosity, and Ohm's law is written as [20] $\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{j}-\lambda\nabla^2\mathbf{j}$.) The important role of the current diffusivity was found there: Below the critical beta value against the global MHD mode, the microscopic interchange mode is destabilized through the current diffusivity, not by the resistivity.

In this Letter we derive the anomalous transport coefficient in the high-aspect-ratio stellarator with magnetic shear and magnetic hill (such as a heliotron device), based on the microscopic current-diffusive interchange mode (abbreviated by "$\lambda$ mode"). We find $\chi_{\rm eff}\propto(d\beta/d\rho)^{3/2}\delta_s^2 v_A/R$ ($\beta$ is the ratio of plasma pressure to magnetic pressure; $\rho$, normalized minor radius $r/a$; $a$, plasma minor radius; $\delta_s$, collisionless skin depth; $v_A$, Alfvén velocity; $R$, major radius). This result is consistent with the experimental knowledge (I)-(IV).

We use a model equation based on the reduced set of equations for stellarators [21]. The cylindrical model [coordinates $(r,\theta,z)$] is employed in order to look for analytical insight for the high-aspect-ratio limit. We also consider the case of zero equilibrium current. The model equations consist of Ohm's law, the equation of motion, and the energy balance equation, as are explicitly given in Ref. [19]. We use the picture of mean-field theory; we analyze the growth rate and mode structure of the $\lambda$ mode by keeping $\chi$, $\nu$, $\lambda$, and $\eta$. The instability-driven transport coefficient (derived by the mixing-length theory) is equated to the one which is used for stability analysis.

We solve the eigenvalue equation by Fourier transformation. The parallel derivative is approximated by $k_\parallel=k_\theta sx$ [$k_\theta=m/\rho_1$, $x=\rho-\rho_1$, $m$ poloidal mode number, $\rho_1$ rational surface, $s=\rho_1\iota'(\rho_1)$, and $\iota$ rotational transform], since we study the microscopic mode which is localized to the rational surface. The variable is changed from $x$ to $k$ as

$$u(x)=\exp[\gamma t+im\theta-im'z/R]\int u(k)\exp(ikx)dk\ ,$$

where $u$ is the perturbed stream function, $\gamma$ is the growth rate, and $m'$ is the toroidal mode number. Eliminating the current and pressure perturbations from the set of equations, and assuming an electrostatic perturbation, we have the eigenvalue equation for $\gamma$ in the $k$ space as [19]

 

$$\frac{1}{L^2}\frac{\partial}{\partial k}\frac{1}{\bar{\eta}+\bar{\lambda}k_\perp^2}\frac{\partial}{\partial k}u+\frac{D_0k_\theta^2}{\bar{\gamma}+\bar{\chi}k_\perp^2}u-(\bar{\gamma}k_\perp^2+\bar{\nu}k_\perp^4)u=0\,, \tag{1}$$

where $k_\perp^2=k_\theta^2+k^2$, $1/L=k_\theta s$, $D_0=-\beta_0\Omega' p'_{\rm eq}/2\varepsilon^2$, $p_{\rm eq}=p_0(\rho)/p_0(0)$ ($p_0$ being the equilibrium pressure), $\varepsilon=a/R$, and $\Omega'$ is approximately given as $\Omega'=\varepsilon^2(N/l)\rho^{-2}(\iota\rho^4)$ ($N$ is the toroidal pitch number and $l$ is the multipolarity). The term $D_0$ denotes the drive by the pressure gradient with bad curvature. The transport coefficients are normalized as $\bar{\chi}=\chi\tau_{Ap}/a^2$, $\bar{\eta}=\eta\tau_{Ap}/\mu_0 a^2$, $\bar{\nu}=\nu\tau_{Ap}/a^2$, $\bar{\lambda}=\lambda\tau_{Ap}/\mu_0a^4$, and the time is normalized as $\bar{\gamma}=\gamma\tau_{Ap}$, where $\tau_{Ap}=R(\mu_0 m_i n_i)^{1/2}/B_0$, $n_i$ is the ion density, $m_i$ is the ion mass, and $B_0$ is the equilibrium magnetic field.

Equation (1) is solved by the Rayleigh-Ritz method. Writing Eq. (1) as $\mathcal{L}u=0$, the functional $\mathcal{R}[u]$ is defined as $\mathcal{R}[u]=\int_{-\infty}^{\infty}u\,\mathcal{L}u\,dk/\int_{-\infty}^{\infty}u^2dk$. The test function $u=\exp(-a^2k^2/2)$ is employed. The equations $\mathcal{R}[u]=0$ and $\partial\mathcal{R}[u]/\partial a=0$ determine the growth rate $\gamma$ and $a$. The value $a$ is the typical radial extent of the mode. The transport coefficient $\chi$ is given by use of the mixing-length model [15] as

$$\bar{\chi}=\bar{\gamma}a^2\,, \tag{2}$$

where a numerical coefficient of order unity is undetermined. The Rayleigh quotient $\mathcal{R}$ is obtained as

$$\mathcal{R}=-s^2\bar{\lambda}^{-1}a^2\{y^2-2y^3\exp(y^2)\,\mathrm{erfc}(y)\}+2D_0k_\theta^2\zeta\exp(\zeta^2)\,\mathrm{erfc}(\zeta)-\bar{\gamma}k_\theta^2\{1+1/2y^2\}-\bar{\nu}k_\theta^4\{1+1/y^2+3/4y^4\}\,, \tag{3}$$

where $y=a^2k_\theta^2$, $\zeta=a^2(\bar{\gamma}/\bar{\chi}+k_\theta^2)$, and $\mathrm{erfc}(y)=\int_y^\infty\exp(-v^2)dv$. (The resistivity contribution is small if $\bar{\eta}<\bar{\lambda}k_\theta^2$. This condition is satisfied, as shown *a posteriori*, and $\eta$ is neglected.) In order to obtain physics insight, we obtain the analytic expression. In the following, we assume that $\bar{\chi}\simeq\bar{\nu}$, since the electrostatic $\mathbf{E}\times\mathbf{B}$ transport is studied. [It is straightforward to study the general case of arbitrary ratio of $\nu/\chi$ (Prandtl number), but this does not change the result qualitatively.]

In the large-$ak_\theta$ limit, the asymptotic limit of the erfc function is used. Taking the leading term in $ak_\theta$ (note $\bar{\chi}=\bar{\nu}=\bar{\gamma}a^2$), the eigenvalue equation $\mathcal{R}[u]=\partial\mathcal{R}[u]/\partial a=0$ gives the growth rate and the radial extent $a$ of the fast interchange mode [22] as $\bar{\gamma}=D_0^{1/2}/(ak_\theta)^2$ and $a^2=[\bar{\lambda}(\bar{\gamma}+2\bar{\nu}k_\theta^2)]^{1/2}/s$. For the small-$ak_\theta$ limit, the Taylor expansion of $\mathcal{R}$ is used. The first-order term is written as $\bar{\gamma}\simeq(\sqrt{8}/5)ak_\theta D_0^{1/2}$. From these results, the largest growth rate is given for the poloidal mode number satisfying $k_\theta a\sim1$. For such a mode, we have the estimate

$$\bar{\gamma}\simeq D_0^{1/2}\,, \tag{4a}$$

$$a^2\simeq s^{-1}(3\bar{\lambda}D_0^{1/2})^{1/2}\,. \tag{4b}$$

Substituting Eq. (4) into Eq. (2), we have

$$\bar{\chi}=\frac{3}{s^2}\frac{\bar{\lambda}}{\bar{\chi}}D_0^{3/2}\,. \tag{5}$$

The value of $\lambda$ is related to the electron viscosity $\bar{\mu}_e$ as [20] $\bar{\lambda}/\bar{\mu}_e=(\delta_s/a)^2$. Since the relation $\bar{\mu}_e/\bar{\chi}\sim1$ holds for electrostatic- and magnetic-turbulence-driven transport [23], we use the relation $\bar{\lambda}/\bar{\chi}\sim(\delta_s/a)^2$. Noting the normalization, the explicit form of $\chi$ is finally obtained as

$$\chi=F(\rho)\{d\beta/d\rho\}^{3/2}\delta_s^2v_AR^{-1}\,, \tag{6a}$$

where $F(\rho)$ is the geometry-dependent numerical coefficient

$$F(\rho)=\frac{3}{s^2}\left\{\frac{N}{2l}\frac{1}{\rho^2}\frac{d}{d\rho}(\iota\rho^4)\right\}^{3/2}\,. \tag{6b}$$

The ratio between the relative amplitude of density and potential fluctuations, $\tilde{n}/n$ and $e\tilde{\phi}/T$, can be derived from Eq. (4). Since the convective change dominates in $n$, we have the relation $\tilde{n}/n=(\omega_*/\gamma)e\tilde{\phi}/T$, where $\omega_*$ is the drift frequency, $Tk_\theta\kappa/eB$ ($\kappa=|\nabla n/n|$ and we assume that $T_e=T_i$). Using the condition $k_\theta a\simeq1$ and the expressions for $\gamma$ and $a$, we have

$$\tilde{n}/n\simeq[3.1sD_0^{-1}\kappa R\beta(a)]e\tilde{\phi}/T\,. \tag{7}$$

This gives the result that the density fluctuation is usually smaller than the potential fluctuation.

We now discuss what is predicted from this model, comparing to the experimental results (I)–(IV).

First, the dimensional dependence of $\chi$ is such that $[\chi]\propto[T]^{1.5}/[R][B]^2$ and is independent of that of density. Equation (6) predicts $\chi$ of the experimental range (see Fig. 1). Second, the point-model analysis gives the energy transport scaling law as

$$\tau_E=A^{0.2}B^{0.8}n^{0.6}a^2RP^{-0.6}\langle F\rangle^{0.4}\,, \tag{8}$$

where $A$ is the ion mass ratio, $P$ is the heating power, and $\langle F\rangle$ is the average of $F$ near the boundary [24]. A weak but positive dependence on the mass ratio is obtained. We also find that the improvement of the confinement by increasing the shear ($s^{-2}$ term in $F$) is almost offset by the increment of the magnetic hill [$\{N(\rho^4\iota)'\}^{3/2}\rho^{-3}$ term in $F$]. This result explains the fact that, from the comparison between different devices, $\tau_E$ seems to depend weakly on the (rotational transform)/shear. $\langle F\rangle^{0.4}$ depends weakly on geometrical parameters. The predicted indices to $B$, $n$, $a$, $R$, and $P$, as a whole, are consistent with the scaling law [5].

Third, the formula $\chi$ includes the radial dependence $(\beta'/n)^{3/2}$, not $T^{3/2}$, and predicts a large transport near the edge. Since the pressure gradient is substantial near the edge (even though the pressure itself must be small) and $n(r)/n(0)$ is decreasing towards the edge, the anomalous transport can be large near the edge. With this radial dependence and that of $F(\rho)$, $\chi$ is larger near the edge, as

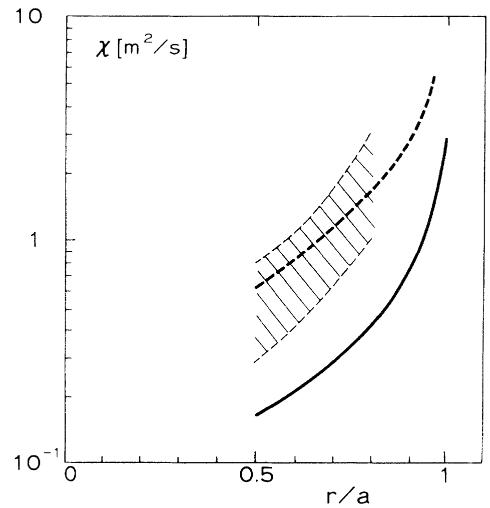


FIG. 1. Example of the prediction of Eq. (6) in the model of Heliotron E [$\bar\iota(\rho)=\bar\iota_0+1.6\bar\iota_1\rho^2$, $\varepsilon=0.1$, $B=2$ T, $T(0)=500$ eV, $n(0)=5\times10^{19}$ m$^{-3}$]. Profiles are chosen such that $p_{\rm eq}(\rho)=1+\Delta-\rho^2$ and $n(\rho)/n(0)=(1+\Delta-\rho^2)^{1/2}$ ($\Delta=0.05$). The thick dashed line indicates 4× formula (6). The shaded region shows the range of experimental data, which is quoted from Refs. [18,25].

is shown in Fig. 1.

Fourth, the heat-pulse propagation time is faster than $\tau_E$. For the case where $|\nabla T/T|\gg|\nabla n/n|$ holds, we obtain a simple relation. Writing the heat flux $q=q_0+\tilde q$ and $\nabla T=\nabla T_0+\nabla\tilde T$, we have $\tilde q=2.5\chi_{\rm eff}\nabla\tilde T$, where $\chi_{\rm eff}$ is the ratio between $q_0$ and $|\nabla T_0|$ (i.e., $\chi$ in this Letter). From this relation, we see that the heat transport coefficient, which is predicted for the heat-pulse propagation, $\chi_{\rm HP}$, satisfies the relation

$$\chi_{\rm HP}=2.5\chi_{\rm eff}\,. \tag{9}$$

Fifth, the relative perturbation of density is smaller than that of the potential. We have $D_0\sim 60\kappa_p a\beta(0)$ and $s\sim 4$ for the Heliotron-E plasma [4], which gives $\tilde n/n\sim[2(\kappa/\kappa_p)^2\beta(a)/\beta(0)]e\tilde\phi/T$ ($\kappa_p=|\nabla p/p|$). The value of the expression in square brackets is of the order of one-tenth. Fluctuation measurements in high-power-heating experiments have shown that $\tilde n/n$ is smaller than $e\tilde\phi/T$ [26,27], confirming our model. This relation also suggests that $(\tilde n/n)/(e\tilde\phi/T)$ increases as the pressure profile becomes broader.

These predictions are consistent with experimental results including (I)-(IV).

In summary, we have developed a new model for the anomalous transport in the toroidal helical plasma with magnetic hill and magnetic shear and presented an analytic formula. The microscopic current-diffusive interchange mode ($\lambda$ mode) is analyzed by keeping the transport coefficients $\chi$, $\nu$, and $\lambda$. A mixing-length estimate is used to derive the transport coefficient from the mode growth rate and structure. Mean-field theory is employed so that the obtained transport coefficient is equated with the given value of $\chi$. By this theoretical analysis we derived the formula of the anomalous transport coefficient. The mode analysis gives that the normalized density perturbation is usually smaller than the normalized potential fluctuation. The comparison with experimental results shows that the derived formula recovers the scaling law, radial shape of $\chi$, difference between $\chi_{\rm HP}$ and $\chi_{\rm eff}$, and the relation between the density and potential perturbations. This formula also explains the weak effects of ion mass and rotational transform on $\tau_E$, which are observed in experiments.

It should be noted that Eq. (6) is derived apart from a numerical coefficient of the order of unity. The previous analyses on the resistive interchange mode by the two-point renormalization method have shown a factor of 5 enhancement over the mixing-length estimate [11,13]. Equation (6) will be changed by a factor like that, but the physical dependences of $\chi$ will not be altered.

We would like to note the approximations in relating $\chi$ in Eq. (1) to the global transport coefficient. The coefficients $\chi k_\perp^2$, $\mu k_\perp^2$, and $\lambda k_\perp^2$ are introduced for the small-scale perturbation in Eq. (1). They affect the decorrelation rates of the perturbation part of the energy, momentum, and current. Following the Dupree renormalization [17], the terms $\chi$, $\mu$, and $\lambda$ are treated as independent of $k_\perp$, and are equated to the values which are obtained by the stability analysis. A renormalization study on $\chi$ has been made for instance in Ref. [11]. The expression of $\chi$, Eq. (6), which is the approximation in the mixing-length theory, holds for the global transport process.

Recent theoretical study has shown that subcritical turbulence can be self-sustaining in a shear-stabilized plasma [28]. Our study shows that this is also the case for high-aspect-ratio torsatron-heliotron plasmas through enhancement of the current diffusivity. In this Letter, the analytical expression of the transport coefficient is obtained as well.

The importance of the current diffusivity is shown. The resistivity is negligible if $\bar\eta<\bar\lambda k_\theta^2$ holds. This condition is rewritten by using the result for $\alpha$ ($\simeq 1/k_\theta$) as $\bar\chi/\bar\eta>(a/\delta_s)^2\bar\eta_s^{-2}D_0^{1/2}$. This condition is usually satisfied for experimental plasmas for which transport analyses are applied.

We compare Eq. (6) to the formula derived by Ohkawa for magnetic turbulence [29]. Compared to the Ohkawa formula, $\chi\sim\delta_s^2 v_e/R$, Eq. (6) has an additional dependence on $\beta$. In our model, the current diffusion is proportional to $\delta_s^2$, so that a similar dependence on $\delta_s^2$ is obtained, though the perturbation is assumed to be electrostatic.

The stellarator expansion is used here to study the case of the high-aspect-ratio limit. The important role of the ballooning mode has been pointed out [30] in the system with lower aspect ratio, for which three-dimensional calculations are required. Our study suggests the importance of the current diffusivity in such a case, and future study is necessary.

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