Propagation Rate of Growing Interfaces in Stirred Fluids

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The overall propagation rate of growing interfaces subject to weak random stirring is investigated, based on a variant of the Kardar-Parisi-Zhang equation. Scaling analysis, supported by computations, indicates that random stirring increases the overall propagation rate in proportion to the $\frac{4}{3}$ power of the ratio of stirring rate to growth rate. The generally assumed quadratic dependence on this ratio is valid only for special cases such as periodic flow. On this basis, a modified turbulent-flame speed formula is proposed. Implications concerning interpretation of measurements are noted.

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The structure of stochastically growing interfaces, as characterized by the Kardar-Parisi-Zhang (KPZ) equation and related formulations [1,2], has been found to exhibit a rich scaling phenomenology. Analyses of this phenomenology have focused on the time regime corresponding to transient development. A question that has received less attention is the parametric dependence of the overall propagation rate during the statistically steady propagation that follows transient relaxation.

The latter question has, however, been the subject of long-standing scrutiny in the engineering literature with reference to the overall burning rate of flames propagating through turbulent gases. In particular, a dynamical equation has been formulated [3] that is closely related to the KPZ equation. Adopting KPZ notation, that "flame propagation" equation may be written as

$$\frac{\partial h}{\partial t}+\mathbf{v}\cdot\nabla h=\lambda[1+(\nabla h)^2]^{1/2}+u\,. \tag{1}$$

Here, $h(\mathbf{x},t)$ is the interface height as a function of location $\mathbf{x}$ transverse to the direction of overall propagation, ∇ is the gradient with respect to $\mathbf{x}$, and u and $\mathbf{v}$ are the longitudinal and transverse components, respectively, of the flow field. Ignoring overhangs, h is assumed single valued. As in the original KPZ formulation, u drives fluctuations of h, $\mathbf{v}$ redistributes the fluctuations transversely, and the interface growth process, governed by the constant parameter λ (the "laminar flame speed"), serves to damp fluctuations.

It is assumed that the interface growth process is dynamically passive, so that the flow field can be regarded as prescribed. This assumption ignores thermal expansion and related aspects of real flames that cause u and $\mathbf{v}$ to depend on h.

Expanding the square root based on a small-gradient approximation, consistent with the assumed single valuedness of h, the leading term is removed by the reparametrization $h\rightarrow h+\lambda t$. The next term corresponds to the nonlinear term of the KPZ equation. Full correspondence to KPZ is obtained if u is regarded as a noise term and if the transverse redistribution term is replaced [4] by a diffusion term of the form $-\nu\nabla^2 h$, appropriate on length and time scales large compared to those governing fluctuations of $\mathbf{v}$. The latter replacement is not explicitly implemented here, so this remark serves merely to refine the analogy between KPZ and Eq. (1).

With the stated reparametrization, $\langle\partial h/\partial t\rangle$ is the increase in the propagation rate due to stirring. The dependence of this quantity on properties of the flow field (henceforth assumed isotropic) is investigated by scaling analysis of the approximate equation

$$\frac{\partial h}{\partial t}+\mathbf{v}\cdot\nabla h=\tfrac{1}{2}\lambda(\nabla h)^2+u\,, \tag{2}$$

valid in the limit of weak stirring. In this limit, geometrical considerations [3,5] give

$$\langle\partial h/\partial t\rangle\approx\tfrac{1}{2}\lambda\langle(\nabla h)^2\rangle\,, \tag{3}$$

where the right-hand side is λ times the surface area of the wrinkled interface per unit projected transverse area.

Equation (2) has previously [3,5] been analyzed by integrating the lowest approximation to obtain $h_0=\int u\,dt$ and substituting this into the right-hand side of Eq. (3). The postulate $\nabla h_0\sim u'/\lambda$, where u' is the root-mean-square velocity fluctuation of the flow field, yields $\langle\partial h/\partial t\rangle\sim\lambda(u'/\lambda)^2$. This result has been the genesis of various analyses of turbulent combustion phenomena [6]. In particular, it is the conceptual basis, and a limiting case, of a renormalization group analysis that yielded the first systematic theory of flame propagation through fully developed turbulence [7].

Here, $\langle\partial h/\partial t\rangle$ is estimated without postulating the scaling of ∇h_0 by applying "nonequilibrium Flory theory" [2] to Eq. (2). Taking the gradient of Eq. (2) and rearranging terms, it can be rewritten as

$$\frac{\partial\mathbf{q}}{\partial t}=-\nabla(\mathbf{v}\cdot\mathbf{q})+\tfrac{1}{2}\lambda\nabla q^2+\nabla u\,, \tag{4}$$

where $\mathbf{q}=\nabla h$. As a result of isotropy, u and $\mathbf{v}$ are regarded as noises with a common correlation length, denoted ξ, and with correlation time thus of order ξ/u'. In the absence of noise, $\mathbf{q}$ is identically zero and an initially planar interface remains planar. It advances a distance ξ in a time interval ξ/λ, which is shorter than the noise correlation time ξ/u' for $u' \ll \lambda$ (the weak-stirring regime). Thus, the interface sweeps through the flow faster than the characteristic time for the flow itself to evolve, so the flow may be regarded as frozen, i.e., the t dependences of u and $\mathbf{v}$ can be ignored. Furthermore, the effective correlation time for the influence of the noise on the interface is ξ/λ rather than ξ/u'.

Equation (4) is regarded as an initial-value problem for $\mathbf{q}(\mathbf{x},t)$ with $\mathbf{q}(\mathbf{x},0)=0$. The evolution of the magnitude of $\mathbf{q}$ and of the length scale characterizing its spatial variation is analyzed in order to estimate the steady-state value of q^2 at which growth and decay mechanisms are in balance. The respective terms on the right-hand side of Eq. (4) govern the transverse redistribution of $\mathbf{q}$, its decay, and its generation by stochastic fluctuations, analogous to the discussion in Ref. [1]. The only nonvanishing term initially is ∇u. During a time interval of the order of the effective correlation time ξ/λ, the growth of $\mathbf{q}$ induced by this term may be regarded as deterministic. Estimating ∇u as u'/ξ, this gives $|\mathbf{q}| \sim u'/\lambda$ and thus $q^2 \sim (u'/\lambda)^2$ at time ξ/λ.

If this expression for q^2 is substituted into Eq. (3), it yields the previously derived quadratic dependence of $\langle \partial h/\partial t \rangle$ on u'. This result would be correct if the flow field were periodic rather than random, because then there would be no dynamical time scale greater than the time ξ/λ for the interface to sweep forward one period ξ. The correctness of the quadratic dependence for periodic flows has been demonstrated [8], but it has not been recognized previously that periodic flow is an atypical case because it precludes the further time development of interface fluctuations. It is evident that the required balance between generation and decay of q^2 is not yet achieved because the decay term of Eq. (4) is of order $u'^2/\lambda\xi$ at time ξ/λ, smaller by a factor u'/λ than the source term at that time. The further growth of $\mathbf{q}$ subsequent to time ξ/λ must be considered to obtain the required balance.

This subsequent growth reflects the stochastic nature of the flow field. Regarding the influence of the growth term as a sequence of independent increments with time step of order ξ/λ, $\mathbf{q}$ at time $t \gg \xi/\lambda$ is of order $[t/(\xi/\lambda)]^{1/2}$ times its value u'/λ at time ξ/λ. The growth rate is estimated by dividing by t, giving $\partial q/\partial t \sim u'/(\lambda\xi t)^{1/2}$.

Growth continues until a time t^* at which it is balanced by decay. The decay term at given t is of order $\lambda q^2/l$, where l is a length scale characterizing the transverse variation of $\mathbf{q}$. Since l appears in the denominator, the decay mechanism is least effective at large l, so the largest values of $\mathbf{q}$ will be associated with the largest available length scale. The length scale introduced by the ∇u term in Eq. (4) is ξ. It is demonstrated shortly that the length scale associated with the transverse redistribution term is smaller than ξ, so this term is irrelevant.

Balance is invoked by equating the growth rate $\partial \mathbf{q}/\partial t$ to the decay rate based on the length scale $l \sim \xi$. This gives a balance time $t^* \sim (\lambda/u')^{2/3}(\xi/\lambda)$, and finally $q^2 \sim (u'/\lambda)^{4/3}$ at this time.

Substitution of this result into Eq. (3) yields an increase in the propagation rate proportional to $(u'/\lambda)^{4/3}$ in the weak-stirring limit $u' \ll \lambda$. The characteristic time scale t^* for interface wrinkling is smaller than the flow-field correlation time ξ/u' by a factor of order $(u'/\lambda)^{1/3}$. Relaxation of transients occurs over a distance λt^*, which exceeds the correlation length ξ by a factor of order $(\lambda/u')^{2/3}$.

The irrelevance of the transverse redistribution term is verified self-consistently. An upper bound on transverse displacements induced by this term is given by t^* times the characteristic magnitude u' of $\mathbf{v}$, yielding displacements of order $(u'/\lambda)^{1/3}\xi$. This is smaller than ξ, confirming the claimed irrelevance. A direct estimate of the characteristic transverse displacement over a time t^*, using the same reasoning applied to the growth of $\mathbf{q}$, gives $(u'/\lambda)^{2/3}\xi$, smaller than the bound.

These results suggest a modification of the renormalization group analysis of the propagation rate u_T in fully developed turbulence [7]. A general form governing the weak-stirring limit is $u_T = \lambda + c\lambda(u'/\lambda)^p$, where $p=\frac{4}{3}$ according to the present analysis, but the value $p=2$ was adopted previously. (c is a numerical coefficient.) The renormalization group analysis obtains the cumulative effect of broadband stirring by successive iterations of the equation $u_T^{(i)} = u_T^{(i-1)} + c u_T^{(i-1)}(u'^{(i)}/u_T^{(i-1)})^p$, regarding the previous iterate as the analog of λ for determination of the next iterate, with $u_T^{(0)} = \lambda$. ($u'^{(i)}$ is the component of the velocity fluctuation in the ith wave-number band.) The formal solution for self-similar flow is

$$u_T = \lambda \exp[c(u'/u_T)^p]\,. \tag{5}$$

For small u'/λ, Eq. (5) yields the aforementioned weak-stirring result. For large u'/λ (corresponding to strong turbulence), Eq. (5) yields the scaling solution [9] $u_T \sim u'$ multiplied by the factor $(\ln u')^{-1/p}$.

It is not self-evident that the previous [7] analytical justification of the formal derivation for the case $p=2$ remains valid for $p=\frac{4}{3}$, due to the introduction of the new time scale t^* in the weak-stirring analysis. Nevertheless, Eq. (5) with $p=\frac{4}{3}$ has the practical advantage that it yields the correct scaling in both limits.

The time scale t^* for interface wrinkling impacts the experimental realization of steady-state, spatially homogeneous propagation in weakly stirred fluids. Namely, the longitudinal dimension of the experiment must exceed $\lambda t^* \sim (\lambda/u')^{2/3}\xi$, where ξ is the "integral scale" in fluid-mechanical terminology. Experimental configurations smaller than this are properly interpreted on the basis of transient analysis consistent with the approach outlined here, rather than on the basis of steady-state propagation theory.

A general caveat concerning the interpretation of experimental data based on the present analysis, including Eq. (5) with either value of p, is that thermal expansion and related effects typically associated with the passage of a chemical front through a medium have been omitted. However, this should not invalidate the qualitative considerations concerning the interpretation of experiments.

The scalings derived here for u_T and t^*, valid for spatial dimension $d \geq 2$, are confirmed by numerical simulations implemented in two and three dimensions. The computations involve propagation of an initially linear interface with interface growth rate λ through an orthogonal (square or cubic) lattice of vortices with lattice spacing ξ. The flow is steady in all cases, consistent with the earlier observation that any time evolution of the flow is irrelevant in the weak-stirring limit.

In the two-dimensional computations, the stream function of each vortex, referenced to the vortex center, is of the form

$$\psi = \begin{cases} \pm 0.556 u' \xi (3\rho^4 - 8\rho^3 + 6\rho^2 - 1), & 0 \leq \rho \leq 1, \\ 0, & \rho > 1, \end{cases}$$

corresponding to pure swirl flow specified by [10] $v_\theta = -d\psi/dr$. (Here, $\rho = 2r/\xi$.) The polynomial expression for $0 \leq \rho \leq 1$ is the lowest-order polynomial that gives a finite-range vortex whose flow field is everywhere continuous and differentiable. The coefficient 0.556 is obtained by integrating v_θ^2 over the unit cell of the lattice and setting the result equal to u'^2.

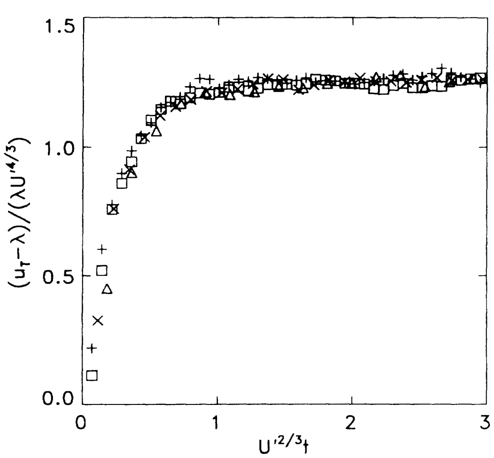

FIG. 1. Time profiles of simulated propagation with interface growth rate λ through a square lattice of vortices with randomly chosen signs. Each time profile is an average of ten realizations for given $U' = u'/\lambda$, where u' is the root-mean-square velocity fluctuation. The width of the computational domain is 256ξ, with periodic boundary conditions applied in the transverse direction. (ξ is the lattice spacing.) The scaled propagation-rate increment $(u_T - \lambda)/\lambda U'^{4/3}$ is plotted vs scaled time $U'^{2/3}t$ for $U' = 0.00241$ (+), 0.00482 (×), 0.00964 (△), and 0.01928 (□), where u_T is the overall propagation rate and time t is expressed in units of ξ/λ.

Each simulated realization involves a flow field consisting of a sum of stream functions of this form, referenced to the respective vortex centers on the square lattice, in conjunction with a rule for choosing the sign of each vortex. This formulation assures that the flow field generated by the lattice of vortices is everywhere continuous and differentiable for any pattern of vortex signs. It also allows the construction of periodic and random flow fields within a common framework.

A periodic case is obtained by assigning a checkerboard pattern of signs to the vortex stream functions. Computations for this case yield quadratic dependence of the propagation-rate increment $u_T - \lambda$ on u' for small u'/λ. Consistent with the analysis [3,5] leading to Eq. (3), u_T/λ is operationally defined as the mean interface arclength divided by the transverse span of the computational domain.

A random case is obtained by independently choosing the sign of each vortex, with equal probability of either sign. Results of computer simulations encompassing a factor of 8 variation of the ratio $U' \equiv u'/\lambda$ are shown in Fig. 1. The collapse of plotted values of $(u_T - \lambda)/\lambda U'^{4/3}$ vs $U'^{2/3}t$ verifies the scalings of $u_T - \lambda$ and t^*.

The scalings are likewise verified in three dimensions, based on a cubic lattice of vortices. Each vortex has the same radial profile of swirl velocity as in the two-dimensional case, modulated axially by a polynomial weighting $2 - 6s^2 + 4s^3$ for $0 \leq s \leq 1$. Here, $s = 2|z|/\xi$ is the scaled coordinate in the direction z corresponding to the axis of the vortex, referenced to the center of the cubic cell. The axial direction is assigned to be either transverse coordinate of the lattice with equal probability. The sign of the vortex with respect to the selected coordi-

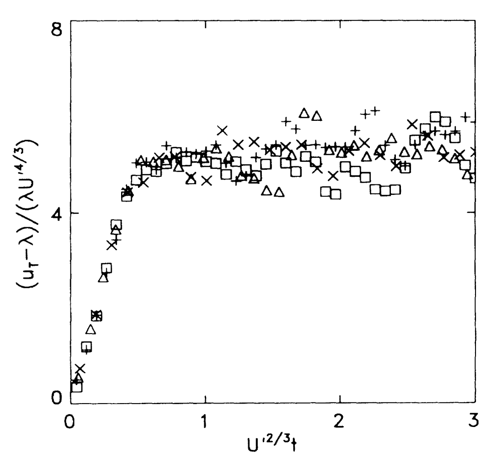

FIG. 2. Time profiles of simulated propagation through a cubic lattice of vortices, each with orientation randomly chosen in one of four directions corresponding to the transverse axes. Each time profile is an average of five realizations for given U'. The width of the computational domain is 2ξ. Profiles are plotted as in Fig. 1 for $U' = 0.0018$ (+), 0.0036 (×), 0.0072 (△), and 0.0144 (□).

nate is likewise assigned randomly. In this case, u_T/λ is operationally defined as the mean interface area divided by the transverse area of the computational domain. Results of simulations encompassing a factor of 8 variation of $U' \equiv u'/\lambda$ are shown in Fig. 2.

Rigorous proof of the $u'^{4/3}$ scaling of $u_T - \lambda$ for $u' \ll \lambda$ is an open question. No rigorous proof yet exists of the weaker result that u_T is finite, i.e., that statistically steady propagation is achieved. The problem of bounding u_T is analogous to the recently solved problem of bounding the effective diffusivity for turbulent transport [11], but with the added complication that the field equation for the present case is nonlinear, even in the weak-stirring limit governed by Eq. (2).

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