# Exact Results for Supersymmetric $\sigma$ Models

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We show that the metric and Berry's curvature for the ground states of $N=2$ supersymmetric $\sigma$ models can be computed exactly as one varies the Kahler structure. For the case of $CP^n$ these are related to special solutions of affine Toda equations. This allows us to extract exact results. We find that the ground-state metric is nonsingular as the size of the manifold shrinks to zero, suggesting that 2D quantum field theory makes sense even beyond zero radius. Thus it seems that manifolds with zero size are nonsingular as target spaces for string theory (even when they are not conformal).

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$N=2$ supersymmetric quantum field theories (QFTs) in two dimensions have been recently investigated extensively (an important subset of them, conformal ones, serve as stationary solutions to superstrings). These theories have the important property of being essentially characterized by simple topological data (chiral ring). Using this simple topological formulation it is possible to extract some exact results for the correlation functions of these highly nontrivial QFTs in two dimensions.

In this Letter we discuss some exact nonperturbative results for $N=2$ supersymmetric $\sigma$ models by applying the results developed in [1]. Let us briefly recall the main results in that paper. Consider a two-dimensional QFT with $N=2$ supersymmetry. Let $Q_\alpha^+$ and $Q_\alpha^-$ label the two supersymmetry charges ($\alpha$ denotes the chirality). The ground states $|a\rangle$ of the supersymmetric QFT are characterized by

$$Q_\alpha^\pm |a\rangle = 0$$

(where we take the space to be a circle of length 1 with periodic boundary conditions). Operators $\phi_i$ which commute with $Q^+$,

$$[Q_\alpha^+, \phi_i] = 0\,,$$

are called chiral. The $CPT$ conjugate operators $\bar\phi_i$ commute with $Q^-$ and are called antichiral. There is a one-to-one correspondence between the ground states and the chiral operators (as can be seen by applying a chiral field to a canonical ground state represented by $|1\rangle$, which is always uniquely definable [1], and projecting it to the canonical ground-state subsector). We can thus label the ground states by the labels of chiral fields $|i\rangle$,

$$\phi_i|1\rangle = |i\rangle + Q^+|\psi\rangle\,.$$

Similarly we can use the antichiral operators to label the same ground states, but in a different basis $|\bar j\rangle$ (which is the conjugate to $\langle j|$). The chiral fields form a commutative associative ring,

$$\phi_i\phi_j = C_{ij}^k\phi_k + [Q^+, \Lambda]\,.$$

We identify the action of $\phi_i$ on the chiral fields with the matrix $C_i = C_{ij}^k$. This matrix also represents the action of $\phi_i$ on the ground states:

$$\phi_i|j\rangle = C_{ij}^k|k\rangle + Q^+|\psi\rangle\,.$$

We can use the top components of $\phi_i$ and its conjugate to perturb the action in a supersymmetric way,

$$\delta S = \int d^2\theta\, d^2z\, \delta t^i \phi_i + \text{c.c.}$$

As we perturb the action by changing $t^i$ the ground states change. We introduce the connections

$$A_i = \langle \bar j|\partial_i|k\rangle$$

and its conjugates, which are defined as a function of $t_i, \bar t_i$ and act on the space of ground states. This connection "measures" the way the ground-state subsector varies in the full Hilbert space as we change the couplings [2]. It is easy to see that as we change the ground-state basis $A_i$ transforms as a gauge connection. Consider the convariant derivatives

$$D_i = \partial_i - A_i,\quad \bar D_i = \bar\partial_i - \bar A_i\,.$$

Let $g$ denote the Hermitian metric in the ground state subsector labeled by the chiral fields:

$$g_{i\bar j} = \langle \bar j|i\rangle\,.$$

It is natural to define in addition a "topological" metric $\eta$ which is symmetric and given by

$$\eta_{ij} = \langle i|j\rangle\,.$$

There is a relation between $g$ and $\eta$; the relation follows from the fact that $|\bar i\rangle$ is the $CPT$ conjugate of $|i\rangle$ and we get

$$g^{-1}\eta(g^{-1}\eta)^* = 1\,. \qquad (1)$$

By the above definition it simply follows that $D_i g = \bar D_i g = 0$. The main result of [1] is to derive a set of differential equations which $g$ and $A$ satisfy as functions of couplings $t_i, \bar t_j$. The equations which we will mainly use give us the curvature of $A$ (which is the generalization of the Berry's curvature to degenerate ground states [2]) in terms of the commutator of the chiral and antichiral rings:

$$[D_i,\bar{D}_j]=-[C_i,\bar{C}_j],\quad [D_i,D_j]=[\bar{D}_i,\bar{D}_j]=0.$$

Choosing a holomorphic gauge ($\bar{A}_i=0$) the first equation can be rewritten (using covariant constancy of $g$) as

$$\bar{\partial}_j(g\,\partial_i g^{-1})=[C_i,g(C_j)^\dagger g^{-1}]. \tag{2}$$

This is the main equation we shall use in this paper to solve for $g$.

We now wish to apply this formalism to the supersymmetric $\sigma$ model on a Kahler manifold $M$. In such a case the ground states, and thus the chiral fields, are in one-to-one correspondence with cohomology elements of $M$ [3]. Moreover $\eta$ may be identified with the intersection of cohomology elements. The chiral rings, ignoring instanton corrections, may be identified with the cohomology ring of $M$. It turns out to be possible to find the exact modification to the chiral ring due to instanton corrections [3]. We then simply use (2) [and the constraint (1)] to solve for $g$ (and the ground-state curvature).

Instead of being general let us consider the case of the supersymmetric $CP^{n-1}$ $\sigma$ model which has been studied extensively in the literature. The classical cohomology ring is generated by a single element $x$ [of dimension (1,1)] represented as a bilinear in fermions with the relation

$$x^n=0,$$

which is modified by instanton corrections simply to [3,4]

$$x^n=\beta \tag{3}$$

(reflecting the fact that $2n$ fermions can absorb the zero modes in the presence of instanton) where $-\ln\beta$ is the action for a holomorphic instanton (which wraps the sphere once around the nontrivial two-cycle of $CP^{n-1}$). Note that $\beta$ need not be real as we can add a topological term to the action which gives a phase modification only in the presence of instantons. Since the real part of the action is positive we have $|\beta|\le 1$. The topological metric $\eta_{ij}=\langle x^i x^j\rangle=\delta_{i+j,n-1}$ in the natural bias corresponding to $\int x^{n-1}=1$. Since $x$ represents the Kahler class, the top component of it is related to the action itself. So we can write the action as

$$S=-\ln\beta\int d^2z\,d^2\theta\,x+\text{c.c.} \tag{4}$$

(For a full Landau-Ginzburg description of $CP^{n-1}$ see [5].) Variation of $\beta$ brings down $-x/\beta$, which is represented in the basis of monomials $x^i$ by the matrix

$$C_\beta=-\frac{1}{\beta}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ \beta & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that $x^i$ has (left,right) fermion number $(i,i)$. But fermion number is violated by multiples of $2n$ units due to instanton effects. So we still have a $Z_{2n}$ conservation of chiral fermion number which in particular implies that $g$,

$$g_{i\bar{j}}=\langle\bar{x}^j|x^i\rangle,$$

is diagonal. Define

$$q_i=\ln g_{i\bar{i}}-\frac{2i-n+1}{2n}\ln|\beta|^2,$$

$$z=n\beta^{1/n}.$$

Then Eq. (2) becomes

$$\partial_z\partial_{\bar{z}}q_i+e^{(q_{i+1}-q_i)}-e^{(q_i-q_{i-1})}=0$$

with $q_n$ defined to be the same as $q_0$. This equation is the familiar affine $\hat{A}_{n-1}$ Toda equation. Using (1) and the form of $\eta$ we learn that $q_i+q_{n-1-i}=0$ which reduces the above equations to the $C_m(BC_m)$ Toda equation, where $n=2m(n=2m+1)$.

The metric $g$ is a function of $|\beta|^2$. This is due to the fact that we need to have an equal number of instantons and anti-instantons to get a nonzero contribution to $g$ (otherwise fermion zero modes will kill the contribution). So the above equations become one dimensional. In other words we are looking for radially symmetric solutions of affine Toda equations. The only additional ingredient we have to provide to completely solve the above equations is the boundary condition. We do this near $\beta\sim 0$, where the radius of $CP^{n-1}$ goes to infinity, and we can use semiclassical arguments to find the norm of the states by representing them as harmonic forms. Namely,

$$\langle\bar{x}^r|x^r\rangle=2^{n-1-2r}\int x^r\wedge *x^r$$
$$=\frac{r!}{(n-1-r)!}(-2\ln|\beta|)^{n-1-2r} \tag{5}$$

(here we used that the Kähler form $k=-\ln|\beta|x$ and that $*k^r=[r!/(n-1-r)!]k^{n-1-r}$). However, this ignores the loop corrections; it turns out that the solutions to the Toda equation themselves know about loop corrections. As is well known the only loop correction to the Kähler *class* is the one-loop result [6] which makes the coupling dynamical. (It is conceivable that there may be loop corrections to the composite operators corresponding to $k^r$.) The renormalization group (RG) flow in this case predicts that

$$-\ln\beta = c_1 \ln\mu\,, \tag{6}$$

where $c_1 = n$ is the first Chern class for $CP^{n-1}$ and $\mu$ defines the RG scale. This is reflected in our formalism by the fact that if the size of the one-dimensional circle is $L$ instead of 1 the dimensionless quantity appearing in the solution changes from $\beta^{1/n} \to L\beta^{1/n}$; in other words $\beta^{1/n}$ has secretly the dimensions of mass (which could also be inferred from the fact that $x^n = \beta$ and $x$ is a fermion bilinear). By introducing a mass scale $\mu$ we can thus write the solution as a function of $L\mu\beta_0^{1/n}$, where $\beta_0$ is again dimensionless. This indeed tells us that $\beta_0$ flows with RG scale according to (6). However, it turns out that the one-loop computation has a finite leftover piece which gives a finite quantum correction to the effective coupling $\beta$; in the minimal subtraction scheme we find the correction to be $-\ln\beta \to -\ln\beta - c_1\gamma$, where $\gamma$ is the Euler's constant. So a more accurate semiclassical computation which takes into account loop corrections should replace (5) by

$$\langle \bar{x}^r | x^r \rangle = \frac{r!}{(n-1-k)!}[2(-\ln|\beta| - n\cdot\gamma)]^{n-1-2r}.$$

The equations we get for the case of $CP^1$ and $CP^2$ turn out to have been studied extensively [7,8]. After imposing the reductions discussed above the equation for the $CP^1$ case becomes the special case of the Painleve III equation. Then the only consistent solutions which have no pole as a function of $\beta$ which have a logarithmic $\beta$ dependence can only be of the form (after changing back to physical variables) [7], as $\beta \to 0$,

$$\langle \bar{x} | x \rangle = \frac{1}{2(-\ln|\beta| - 2\gamma)}[1 + O(|\beta|^2 \ln^2|\beta|)]\,.$$

Needless to say we can compute order by order the instanton contributions by solving the differential equation. Indeed, for small $\beta$, $\langle \bar{x}|x\rangle^{-1}$ has an expansion of the form [9]

$$\langle \bar{x}|x\rangle^{-1} = \sum_{n=0}^{\infty} |\beta|^{2n} p_n\,,$$

where $p_n$ is a polynomial in $[-\ln|\beta| - 2\gamma]$ of degree $2n+1$. The coefficients of these polynomials can be computed recursively from the differential equation. (For $n \le 3$ these coefficients are listed in Ref. [9].) The corrections clearly reflect the contribution of $n$ instanton–anti-instanton pairs to the metric (as they have a prefactor of $|\beta|^{2n}$). It would be interesting to compare the $p_n$ from perturbations about the instanton backgrounds. It is also very satisfactory that the form of the loop corrections is *predicted* to be that given by the one-loop term (including the Euler's constant) simply by requiring nonsingularity of the solution as a function of $\beta$. In fact the solution can be continued analytically even past $\beta = 1$ which corresponds to zero radius on the sphere to $\beta \ge 1$ which has no obvious relation to the $\sigma$ model on sphere. We should have expected that we will not encounter any singularities because according to the renormalization-group flow computation (6) in *finite* RG time (i.e., finite mass scale) we come to have a zero radius, and if the theory is sensible at all we should be able to choose any mass scale which would correspond to passing through zero radius. In fact the asymptotic structure to the metric has also been worked out as $\beta \to \infty$, and one finds that

$$\frac{\langle \bar{x}|x\rangle}{\langle 0|0\rangle} = |\beta| \left(1 - \frac{1}{(\pi^2|\beta|)^{1/4}}\exp(-8|\beta|^{1/2})\right) + \cdots\,.$$

The interpretation of this is as follows: The ring $x^2 = \beta$ suggests that for large $\beta$ the field configuration is dominated near $x = \pm\sqrt{\beta}$, which explains the leading term in the above asymptotic behavior. The subleading exponential term suggests that there is tunneling between these configurations by a soliton with mass $8|\beta|^{1/2}$. The behavior of this theory is very similar to that of the Landau-Ginzburg (LG) theory which has the same ring, namely, $W = x^3/\beta - 3x$. The difference is that the behavior as $\beta \to 0$ is different between the two and thus we get a different solution of the Painleve III equation. The $\beta \to \infty$ behavior for the LG theory is the same as the above, with the difference that the coefficient in front of the exponential term is smaller by a factor of 2. In [1] this coefficient was (heuristically) related to the number of particles, and so this suggests that in the $CP^1$ model we have twice as many particles as in the LG theory, which is indeed the case, since $CP^1$ is believed to have [10] one doublet of SU(2) whereas the LG theory has only one particle.

This story can be repeated exactly as before for $CP^2$, where the relevant equation has been studied in [8] with the result that (with $s = 3$, $g_1, g_2 = 0$, $g_3 = 1$ in Kitaev's notation) we get a nonsingular solution, where the boundary condition is again predicted to be the same as (5) with the coefficient in front of $\gamma$ being $n = 3$ the first Chern class of $CP^2$. Again the asymptotic behavior is worked out and one finds a similar behavior, this time predicting the existence of particle of mass $6\sqrt{3}|\beta|^{1/3}$. Moreover the strength of the soliton correction is 3 times larger than the LG theory, $W = x^4/\beta - 4x$, suggesting that we have 3 times as many particles. This is indeed the case, as $CP^2$ is believed to have six particles (3 and $\bar{3}$) [10] where the LG theory is believed to have only 2 (which has been recently confirmed in [11]).

Needless to say we believe that similar behavior will work for general $CP^{n-1}$, the difference being that there the explicit solution to the (reduced) affine Toda equations has not been worked out. However, using physics we can predict what would be a nonsingular solution to the corresponding Toda equations. Moreover the result in appendix B of [1] shows that from $\beta \to \infty$ we can read off the spectrum of masses of particles and this turns out to be $4n\sin(\pi r/n)|\beta|^{1/n}$ as $r$ runs from 1 to $n-1$, in agreement with [10]. One should in principle be able to read off degeneracies as well by a careful study of the strength of the soliton corrections to determine the degeneracy of particles as we did for $CP^1$ and $CP^2$. We can repeat this game for any manifold for which we know the quantum cohomology ring. This is in fact conjectured for Grassmanians in [12] (see also [4]); a simple extension should work for all Hermitian symmetric spaces. At any rate we can for example read off the masases for the Grassmanians.

Probably the most important aspect of this work is that it suggests we can go beyond zero radii for supersymmetrical $\sigma$ models thus hinting that with or without conformal symmetry $\sigma$ models somehow "resolve" singularities of classical geometries. This was connected with the fact that in finite RG time there is flow to the singular geometry and completion of the moduli space *requires* having something beyond the singular point which is what seems to happen. It would be interesting to unravel the geometry beyond zero radius. In some ways this is related to the same phenomenon for Calabi-Yau manifolds where the boundaries of moduli space are in some cases a *finite* distance away from any point, and thus completion of string theory suggests that we can go beyond them, as has been suggested in [13]. Some aspects of our computations are similar to the recent one of Candelas *et al.* [14] in which they computed instanton corrections to the ground-state metric (Zamolodchikov metric) on a threefold quintic. In fact the equation they obtain can be rephrased as the *standard* $A_3$ Toda equation [1] and so is similar to the off-critical equation which happens to be an *affine* Toda equation (for example, for $CP^3$).

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