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Six-Vertex Model, Roughened Surfaces, and an Asymmetric Spin Hamiltonian

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For a particular choice of vertex weights, the two-dimensional six-vertex model can be viewed as a probabilistic cellular automaton. Physically it describes then the surface slope of a two-dimensional solid which grows through deposition. Based on this analogy we predict the large-scale asymptotic behavior of the vertical polarization correlations. The transfer matrix commutes with a nonsymmetric spin Hamiltonian. We diagonalize it using the Bethe ansatz and prove that the dynamical scaling exponent for kinetic roughening is $z=\frac{3}{2}$ in $1+1$ dimensions.

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In models of statistical mechanics we may interpret one of the spatial coordinate axes as time. The transfer matrix in this direction is regarded as the propagator for a time evolution. Mostly this corresponds then to a quantum field theory with one spatial dimension less than the original model. In certain cases also the interpretation as a stochastic time evolution is possible and the corresponding object has been named probabilistic cellular automaton (PCA) [1,2].

The two-dimensional six-vertex model is no exception to the general rule (cf. Fig. 1) [3]. Time runs upwards. The ice rule ensures that lines do not merge, stop, and bend backwards. Therefore we may interpret them as the world lines of particles. The point of interest is that in this way the six-vertex model can be viewed as a one-dimensional driven lattice gas, which corresponds to the slope of a growing surface. Thereby we arrive at novel predictions for the vertical polarization correlations. Conversely we will diagonalize a spin Hamiltonian commuting with the transfer matrix of the six-vertex model through the Bethe ansatz and extract from it the dynamical scaling exponent for (1+1)-dimensional kinetic roughening.

Let $\sigma=(\sigma_1,\ldots,\sigma_N)$ be the spin configuration of one row of the six-vertex model, where $\sigma_j=1$ stands for occupied and $\sigma_j=-1$ for vacant, and let $\langle\sigma|T|\sigma'\rangle$ be the transfer matrix with periodic boundary conditions from row m with spin configuration σ to row $m+1$ with spin configuration σ'. $\langle\sigma|T|\sigma'\rangle$ can be regarded as the transition probability from σ to σ' in one time step provided the normalization condition

$$\sum_{\sigma'}\langle\sigma|T|\sigma'\rangle=g(y),\quad y=\frac{1}{N}\sum_{j=1}^{N}\sigma_j \tag{1}$$

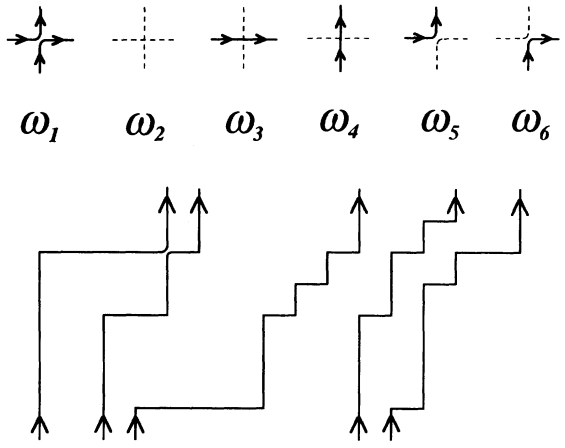

FIG. 1. A six-vertex configuration and the corresponding Boltzmann weights.

holds. Then, in each sector of fixed vertical polarization (particle number) y, T can be normalized simply by dividing through with $g(y)$. [We remark that, in general, the transition probability is given by $\psi^{-1}(\sigma)\langle\sigma|T|\sigma'\rangle \times\psi(\sigma')\lambda_0^{-1}$ with ψ the maximal right eigenvector and λ_0 the maximal eigenvalue of T. As a rule this transition probability is nonlocal and a probabilistic interpretation may not be so useful.]

Equation (1) is ensured by imposing the relation

$$\omega_5\omega_6=(\omega_1-\omega_4)(\omega_2-\omega_3) \tag{2}$$

for the vertex weights (compare with Fig. 1). To verify our claim the simplest way is to note [3,4] that in this case T commutes with a spin Hamiltonian H given by

$$H=-\frac{1}{4}\sum_{j=1}^{N}[\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_{j+1}-1+i\epsilon(\sigma_j^x\sigma_{j+1}^y-\sigma_j^y\sigma_{j+1}^x)], \tag{3}$$

$\boldsymbol{\sigma}_{N+1}=\boldsymbol{\sigma}_1$. Here $\boldsymbol{\sigma}_j=(\sigma_j^x,\sigma_j^y,\sigma_j^z)$ are the Pauli spin matrices at site j satisfying $[\sigma_j^x,\sigma_l^y]=\delta_{jl}2i\sigma_j^z$ plus cyclic permutations and the asymmetry parameter ϵ is given by

$$\epsilon=(\omega_2\omega_4-\omega_1\omega_3)/(\omega_2\omega_4+\omega_1\omega_3), \tag{4}$$

$|\epsilon|\le 1$. $\langle\sigma|T|\sigma'\rangle$ are the matrix elements of the operator T in the σ^z representation. By a straightforward computation, H has in each sector a nondegenerate ground state, which has the eigenvalue zero and gives equal weight to each spin configuration. Since $[T,H]=0$, T and e^{-tH} have the same maximal eigenvectors by the Perron-Frobenius theorem, which implies (1).

For $\epsilon=0$, H is just the isotropic Heisenberg ferromagnet [5–7]. For $\epsilon\neq 0$, H is *nonsymmetric* and has therefore no quantum-mechanical interpretation. However, H is still the generator of a Markov semigroup, a fact to which we will return shortly.

Before continuing our analysis we should clarify which part of the phase diagram of the six-vertex model is singled out by relation (2). Using the standard labeling of vertex weights (cf. Ref. [3] and Fig. 1), relation (2) means that $\Delta\ge 1$ with $\epsilon=0$ corresponding to $\Delta=1$, where $2\Delta=(\omega_1\omega_2+\omega_3\omega_4-\omega_5\omega_6)(\omega_1\omega_2\omega_3\omega_4)^{-1/2}$. Here y is fixed, but without constraint y would stick at either $+1$ or -1 [3]. The free energy as a function of the horizontal field h is convex downwards and has a linear piece [8,9]. Relation (2) implies that h lies inside the interval of phase coexistence.

As can be seen from the Fig. 1, $g(y)^{-1}\langle\sigma|T|\sigma'\rangle$ is the transition probability for particles jumping randomly to the right constrained by exclusion. Because of the conservation law, this is a diffusive system of particles driven by some external force. If we coarsen somewhat in space and time, then the local vertical polarization $u_t(x)$ is governed by a fluctuating equation of conservation type [10],

$$\frac{\partial}{\partial t}u_t(x)+\frac{\partial}{\partial x}\left[j(u_t(x))-\nu\frac{\partial}{\partial x}u_t(x)+J_t(x)\right]=0. \tag{5}$$

The local current is written here as the sum of a systematic current j, a diffusive current $-\nu\partial u_t(x)/\partial x$, and a fluctuating current which is Gaussian white noise with correlations $\langle J_t(x)J_{t'}(x')\rangle=\gamma\delta(t-t')\delta(x-x')$. The last two terms are partly phenomenological. The real microscopic input is the systematic current j. From the transfer matrix we obtain [11]

$$j(y)=2(1+y)/[a(1-y)+(1+y)] \tag{6}$$

with $a=\omega_1(\omega_2-\omega_3)/\omega_2(\omega_1-\omega_4)$. If we are interested only in the fluctuations around a uniform profile, then it suffices to expand $j(y)$ up to second order, higher orders being irrelevant. Equation (5) becomes then the noisy Burgers equation,

$$\frac{\partial}{\partial t}u_t(x)+\frac{\partial}{\partial x}\left[c(y)u_t(x)+\frac{1}{2}\lambda(y)u_t(x)^2-\nu\frac{\partial}{\partial x}u_t(x)+J_t(x)\right]=0, \tag{7}$$

where we have written the fluctuating profile as $y+u_t(x)$ and

$$c(y)=j'(y),\quad \lambda(y)=j''(y). \tag{8}$$

Equivalently, we may introduce height variables h_j through $h_{j+1}-h_j=(\sigma_{j+1}+\sigma_j)/2$. The h dynamics describes then the surface of a two-dimensional solid which grows through deposition of pieces of various lengths at local surface minima. Also evaporation processes are allowed. The phenomenon of interest is kinetic roughening: One starts with a flat surface which roughens as the solid grows. If we substitute $h_t'=u_t$ in Eq. (7), then it goes over to the (1+1)-dimensional Kardar-Parisi-Zhang equation whose (2+1)-dimensional variant describes growing surfaces in three space [12,13].

For $\epsilon\neq 0$ also $\lambda\neq 0$. The dynamic structure function $S(k,t)$ for the noisy Burgers equation is then expected to be of the scaling form [13,14]

$$S(k,t)=\frac{\gamma}{2\nu}e^{ikct}\Phi((\lambda^2\gamma/2\nu)^{1/3}k|t|^{2/3}) \tag{9}$$

for small k and large t (hydrodynamic regime) with the static compressibility $\gamma/2\nu=1-y^2$. We note that the dynamic scaling exponent is $z=\frac{3}{2}$. If $\epsilon=0$, then j is linear and $\lambda=0$. Thus Eq. (7) becomes linear and the exact dynamic structure function is $(\gamma/2\nu)\exp(ikct-\nu k^2|t|)$. In particular, the dynamic scaling exponent is $z=2$.

In conclusion, we arrive at the following prediction for the vertical polarization correlation function for the six-vertex model. We assume (2), $\epsilon\neq 0$, and fix y. Then the Fourier transform in the horizontal direction is given by (9) where t refers to the vertical direction. In particular, along the ray with slope $c(y)^{-1}$ the asymptotic decay is $r^{-2/3}$, with r the relative distance. Along all other directions the decay is exponential with a crossover determined

by (9).

Instead of transferring along one of the lattice axes, as done here, another option would be the diagonal transfer matrix. It can be viewed also as a PCA [15,16] provided the vertex weights satisfy $\omega_5=\omega_2-\omega_3$ and $\omega_6=\omega_1-\omega_4$, which is more restrictive than (2). The symmetric case is studied in Ref. [17].

We may turn our story around and use the exact solution of the six-vertex model with the goal of obtaining the scaling exponent $z=\frac{3}{2}$ and the scaling function Φ. It is less cumbersome to work with the spin Hamiltonian (3) commuting with the six-vertex transfer matrix. In fact, $\langle\sigma|e^{-tH}|\sigma'\rangle$ is again a transition probability although now continuous in time. As can be checked from (3) it already satisfies the normalization condition (1) with $T=\exp(-tH)$ and $g(y)=1$. According to this transition probability, particles jump with rate $(1+\epsilon)/2$ to the right and with rate $(1-\epsilon)/2$ to the left constrained by single-site occupancy. Such a model is known as an asymmetric simple exclusion process in the probabilistic community [18]. Among those studying kinetic roughening it is the single-step model [19]. In the surface interpretation local minima are filled with rate $(1+\epsilon)/2$ and local maxima evaporate with rate $(1-\epsilon)/2$.

The dynamic scaling exponent describes the long-time decay as dominated by the spectral gap of H. If E_N denotes the real part of the first excited state of H, then we expect that $E_N\approx N^{-z}$ for large N. It is known that the Heisenberg Hamiltonian has a spectral gap of the order N^{-2} uniformly in the density [20]. This covers the case $\epsilon=0$. For $\epsilon\neq 0$ we turn to the Bethe diagonalization of H [11]. In a short announcement [21] Dhar states that for the completely asymmetric ($\epsilon=1$) case the Hamiltonian (3) can be diagonalized through the Bethe ansatz and that the spectrum has a gap of order $N^{-3/2}$. No further details have appeared yet.

We label a spin configuration by the location of the up spins, $\mathbf{m}=(m_1,\ldots,m_n)$ with the convention $1\le m_1<m_2<\cdots<m_n\le N$. Let $|\phi\rangle$ be a right eigenstate for H, $H|\phi\rangle=E|\phi\rangle$. It may be expanded in the above basis as

$$|\phi\rangle=\sum_{\mathbf{m}} f_{\mathbf{m}}|\mathbf{m}\rangle. \tag{10}$$

The Bethe ansatz consists of choosing the expansion coefficients as

$$f_{\mathbf{m}}=\sum_{\mathcal{P}}\mathcal{A}(\mathcal{P})\prod_{j=1}^{n}(z_{p(j)})^{m_j}, \tag{11}$$

where $\mathcal{P}=(p(1),\ldots,p(n))$ denotes a permutation of $(1,\ldots,n)$ and the sum is over all permutations. The eigenvalue problem for periodic boundary conditions is solved by (11) provided the complex "wave numbers" z_j, $j=1,\ldots,n$, satisfy the Bethe equations,

$$\begin{aligned}(z_j)^N&=\prod_{j\neq l=1}^{n}W(z_j,z_l),\\ W(z_j,z_l)&=-\frac{2z_j-(1+\epsilon)z_jz_l-(1-\epsilon)}{2z_l-(1+\epsilon)z_jz_l-(1-\epsilon)},\end{aligned} \tag{12}$$

for $j=1,\ldots,n$. The expansion coefficients in (11) are then recursively defined through

$$\mathcal{A}(\ldots,p,q,\ldots)=W(z_p,z_q)\mathcal{A}(\ldots,q,p,\ldots). \tag{13}$$

In general (12) must be solved as a system of n equations. However, for $\epsilon=1$ a drastic simplification occurs: The eigenfunctions become determinants, i.e.,

$$f_{\mathbf{m}}=\det\mathcal{M},\quad \mathcal{M}_{jl}=(z_l)^{m_j}(1-z_l^{-1})^{j-(n+1)/2}. \tag{14}$$

If in addition the system is half filled, $n=N/2$, then the Bethe equations read, with $z_j\equiv(1+Z_j)/2$,

$$[(1+Z_j)(1-Z_j)]^n=-4^n\prod_{l=1}^{n}[(Z_l-1)/(Z_l+1)]\equiv Y, \tag{15}$$

$j=1,\ldots,n$. The corresponding energy is

$$E=\frac{1}{2}\sum_{j=1}^{n}(1-Z_j). \tag{16}$$

To solve (15) we pick a Y and take its nth root. Each Z_j then satisfies a quadratic equation having two solutions. To each choice of n out of $2n$ solutions there correspond an eigenvector and its eigenvalue. Finally for each particular choice we have to determine Y through (15). For the ground state we obtain $Y=0$ and $Z_j=1$ for $j=1,\ldots,n$. For the first excited state we find $|Y^{1/n}|=1+O(1/n)$ and $\arg(Y)=\pi$. The resulting Z_j's lie on a club-shaped curve in the complex plane. To our surprise they are not obtainable through a small perturbation of the ground state. By a careful examination of the complex contour integral, we obtain analytically $E_N\approx(6.509\,17\ldots)N^{-3/2}$ for large N.

Finally, we remark on a connection of the present work with the theory of conformal invariance [22]. It is well known that the symmetric Heisenberg spin Hamiltonian is conformally invariant in the antiferromagnetic regime, which corresponds to reversing the sign in (3). Although the mapping to growth models is lost in this case, the Bethe ansatz solutions remain as before, except for an inversion of the spectrum. Therefore, it could be an ideal test ground for extending conformal field theory to non-Hermitian models. For the half-filled system and $\epsilon=1$ we determine analytically the spectral gap for the "antiferromagnetic" version of (3) [11]. The gap turns out to have a constant imaginary part and a real part of order $1/N$ consistent with the finite-size scaling predictions of symmetric conformal theory [22,23].

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