Asymptotic Uniqueness of the Sliding State for Charge-Density Waves

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Analytic results are presented for the dynamic behavior of sliding charge-density waves (CDW's) and related extended nonlinear systems with randomness. It is shown that, in the limit of long times, a sliding configuration approaches a unique solution. In CDW models, the velocity of this asymptotic solution is periodic in time. These results explain previous numerical observations, simplify further analysis and simulation, and have direct experimental implications.

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There are many physical systems, including interfaces in random media, flux lattices in type-II superconductors, and charge-density waves (CDW's), where many degrees of freedom interact strongly in a random environment. A common problem in such systems is determining the response to an external drive, e.g., current or electric field; this is the problem of collective transport in systems with quenched randomness [1]. A general difficulty in treating random systems is the existence of many metastable solutions, which can complicate both analysis and numerical computations. However, in some systems exhibiting collective transport, the dynamics, though still complex, is simpler than might be expected. I report here analytic results that explain this behavior, under certain conditions, using CDW transport as an example system and summarizing the extension to other systems below.

An incommensurate charge-density wave in a solid can be modeled as an elastic medium that is subject to both spatially varying pinning forces, due to impurities, and a uniform external force, due to an electric field [2–4]. If thermal noise can be neglected, the possible dynamical behaviors of CDW's can be sharply divided into two types: pinned and sliding. For small drive fields, the impurities pin the CDW in one of many static configurations. If the field is then increased above a threshold value, the CDW slides with a nonzero average velocity, thereby contributing to the electric current. The equations of motion for the CDW are nonlinear and describe an extended system with quenched randomness. It therefore would not be surprising for the CDW to exhibit some sort of turbulent behavior in the sliding state, or, at least, that there would be an aperiodic attractor or multiple attractors for the CDW dynamics.

However, as I show in this Letter for a broad class of models describing CDW's and related systems, sliding configurations approach, at long times, a solution that is *unique* up to time translation. Also, the transition between pinned and sliding states is nonhysteretic. These results are generally applicable when the equations of motion are first order in time, phase distortions are described by a single component, and the elastic force between phases on neighboring sites increases with the phase gradient (i.e., the interaction potential is convex). For the periodic pinning potential of the CDW model, the CDW sliding state has a particularly simple temporal behavior, with the CDW velocity at each point being periodic in time, though the spatial behavior is quite complex [2,3,5–7].

These results confirm and extend conjectures based on numerical work for CDW's [5] and have direct experimental and theoretical implications [2,3,8]. An experimental consequence is that, after initial transients, the properties of a CDW measured in the sliding state should be independent of history. This is consistent with experiments in the sliding state [2,3] and is in marked contrast with measurements in the pinned state [2,3,9]. In addition, the observed broadband noise in CDW's must be due to effects not included in the simplest deterministic models, perhaps thermal fluctuations or CDW defects. The uniqueness and periodicity of the sliding state are useful in designing and interpreting numerical simulations of CDW models [5,7]. The analytical treatment of the critical behavior of the CDW depinning transition is also aided by these results. The assumption of uniqueness explicitly underlies previous speculations on the critical behavior by Fisher [8] and increases the possibility of successful perturbative treatment.

The CDW model [4] that will be used as the primary example here is a simplification of the dynamical model due to Sneddon, Cross, and Fisher (SCF). The simplified model reproduces the complex behavior of CDW's, including the depinning transition, mode-locking, and subthreshold hysteresis [2–4]. The CDW configuration in this model is described by the real variables φ_i, which give the CDW distortions at N lattice sites indexed by i. The equations of motion for an overdamped CDW, derived from a potential which is the sum of pinning, drive, and simple elastic terms, are [5,6]

$$\dot{\varphi}_i = \Delta^2 \varphi_i - V_i'(\varphi_i) + F(t)\,, \qquad (1)$$

with Δ^2 being the lattice Laplacian and $\dot{\varphi}_i \equiv (d/dt)\varphi_i$, where t is time. The N pinning potentials V_i, from which the pinning forces $-V_i'$ arise, are each periodic, with period 2π, and are assumed to be continuously differentiable. The average CDW velocity is defined as $v = \lim_{T\to\infty}(NT)^{-1}\sum[\varphi_i(T)-\varphi_i(0)]$ and is positive for constant drive fields $F > F_T^+$, where F_T^+ is the positive threshold field. Note that the elastic potential is convex, with the magnitude of the elastic forces (given by the $\Delta^2\varphi_i$ term) increasing for increasing separation of nearest-neighbor φ_i's.

A very useful constraint on CDW motion is given by a "no-passing" rule [7]. Given a drive F and pinning potentials V_i, consider two initial CDW phase configurations, with one configuration, $\varphi_i^L(0)$, the "lesser" and the other, $\varphi_i^G(0)$, the "greater," with $\varphi_i^G(0) \geq \varphi_i^L(0)$, for all sites i. The no-passing rule states that the greater solution is never "passed" by the lesser at any point or subsequent time, i.e., $\varphi_i^G(t) \geq \varphi_i^L(t)$, for all i and all $t \geq 0$. This can be seen by noting that if passing were to occur, the lesser solution must first approach the greater one. If the two solutions approach each other at some initial crossing site, the pinning and drive forces for the two solutions tend to become equal at that site. The elastic forces due to neighboring sites therefore determine the relative evolution of the two solutions at that site and prevent the lesser solutions from passing the greater (see Fig. 1). This rule relies crucially on the elastic potential being convex.

It follows from the no-passing rule that, for a finite system with given pinning V_i and constant or periodic drive field $F(t)$, all solutions to the equations of motion must have the same long-term average velocity v. Choose any two solutions to the equation of motion. Using the $\varphi_i \to \varphi_i + 2\pi$ invariance of the equations of motion, either solution can be made to be the lesser initially, without changing its velocity. By the no-passing rule, the motion of the lesser solution is bounded by the greater, implying that the lesser solution has a velocity smaller than or equal to that of the greater [10]. Since the choice of the initially lesser configuration is arbitrary, the velocity must be independent of the initial configuration [11]. A corollary is that the threshold field for sliding, F_T^+, is independent of history.

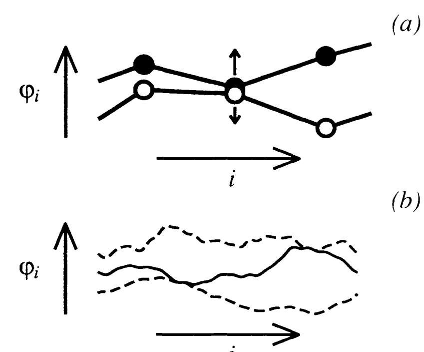

FIG. 1. (a) Two configurations, φ_i^G (solid circles) and φ_i^L (open circles), which are ordered at time t, i.e., $\varphi_i^G(t) \geq \varphi_i^L(t)$, for all i. If the configurations approach each other, local pinning and drive forces become equal. Elastic forces, represented by the arrows, then prevent φ_i^L from exceeding φ_i^G at any site or time, resulting in the "no-passing" rule. (b) The relationship between bounding configurations $\varphi_i^P(\tau^{\pm}(t))$ (dashed lines) and the configuration of an arbitrary solution $\varphi_i(t)$. The bounding of the arbitrary configuration and the subsequent separation of the configurations as they evolve is used to prove asymptotic uniqueness of the sliding state.

I now outline the proof that all sliding solutions to Eq. (1) approach a solution that is unique up to time translation, as $t \to \infty$, given a pinning realization V_i and constant $F > F_T^+$. Details of this proof are described below. It is first necessary to show the existence of a solution where the local velocities are positive at all sites. This solution then gives a set of CDW configurations, indexed by the time. The configuration of an *arbitrary* solution at a given time can be bounded from above and below by configurations chosen from this set. The best upper and lower bounds are those which coincide with the given configuration at one or more points [see Fig. 1(b)]. These bounding configurations are then considered as initial conditions for the equation of motion and the subsequent evolution of these "bounding solutions" can be compared with that of the arbitrary solution. As time increases, the arbitrary solution *separates* from the bounding solutions, due to elastic forces, which determine the relative evolution of two solutions at points of coincidence. The upper and lower bounding configurations can therefore be improved as time increases, with the improved bounds approaching each other as $t \to \infty$. The arbitrary solution therefore approaches the positive velocity solution at long times (for $F > F_T^+$).

The construction of a positive velocity solution relies on a result similar to the no-passing rule: If the local velocities $\dot\varphi_i$ for a solution to Eq. (1) are initially all positive, then they will always be positive (though the velocities approach 0 as $t \to \infty$ for $0 \leq F \leq F_T^+$). By differentiation of the equations of motion Eq. (1),

$$\ddot\varphi_i(t) = \Delta^2\dot\varphi_i(t) + V_i''(\varphi_i(t))\dot\varphi_i(t) . \qquad (2)$$

Suppose that at time $t=0$, all of the velocities are positive, $\dot\varphi_i(0) > 0$, but that there is a time $t^* > 0$, which is the first time at which the velocity at any site is zero. Let j be a site where $\dot\varphi_j(t^*) = 0$. As the velocity on neighboring sites is non-negative, $\Delta^2\dot\varphi_j(t) \geq 0$, which, by Eq. (2), implies that $\ddot\varphi_j|_{t^*} \geq 0$. Consequently, $\dot\varphi_j(t) \leq 0$ in some interval of time preceding t^*, contradicting the definition of t^*. The velocities are therefore always positive.

It is straightforward to construct a solution with initial velocities all positive. Let $\varphi_i^P(t)$ be the solution to the equations of motion for constant $F > 0$, with initial condition $\varphi_i^P(0)$ chosen to be a configuration that is static for $F=0$. By Eq. (1), the local velocities at time $t=0^+$ are then all equal to $F > 0$ and hence positive for all time.

Given constant $F > F_T^+$, let $\varphi_i(t)$ be an arbitrary solution to the equations of motion Eq. (1). One can then define a function $\tau^-(t)$, which is the maximum time at which the positive velocity solution φ_i^P trails $\varphi_i(t)$:

$$\tau^-(t) \equiv \sup[s \,|\, \varphi_i(t) \geq \varphi_i^P(s), \text{ for all } i] . \qquad (3)$$

Similarly, $\tau^+(t)$ is defined as the least time at which φ_i^p is greater than $\varphi_i(t)$:

$$\tau^+(t) \equiv \inf[s|\varphi_i(t) \le \varphi_i^p(s), \text{ for all } i] . \quad (4)$$

The function $\tau^+(t)$ is well defined for all t, since $F > F_T^+$ implies that $\varphi_i^p(s)$ is unbounded for all i. The function $\tau^-(t)$ is well defined after the earliest time at which $\varphi_i(t)$ is greater than the configuration $\varphi_i^p(0)$.

To obtain the desired result, it is necessary to show that the difference in the bounding times, $\Delta\tau \equiv \tau^+(t) - \tau^-(t)$, approaches zero as $t \to \infty$. Given a time $t_0 \ge 0$, consider the two solutions to the equations of motion $\varphi_i(t_0+t')$ and $\varphi_i^p(\tau^+(t_0)+t')$. If one defines the field

$$\psi_i(t') = \varphi_i^p(\tau^+(t_0)+t') - \varphi_i(t_0+t') , \quad (5)$$

then $\psi_i(t') \ge 0$ for all $t' \ge 0$, by the no-passing rule and the definition of $\tau^+(t)$. Actually, a stricter condition holds: $\psi_i(t') > 0$ for all $t' > 0$, unless $\varphi_i^p(\tau^+(t_0)) = \varphi_i(t_0)$ for all i. This can be seen by considering the equations of motion for $\psi_i(t')$ evaluated at a site where $\psi_i = 0$:

$$\dot\psi_i|_{\psi_i=0} = \Delta^2\psi_i . \quad (6)$$

Suppose that at a site i, $\psi_i = 0$. If $\psi_j > 0$ for some neighboring site j, $(d/dt')\psi_i > 0$. It can be seen by repeated differentiation of Eq. (6), that, if the nearest site j where $\psi_j > 0$ is m sites distant, $(d/dt')^m\psi_i > 0$, with all lower-order derivatives zero. As the lowest-order nonzero derivative of $\psi_i(t')$ is positive, $\psi_i(t') > 0$ for $t' > 0$ (once ψ_i is positive, it cannot return to zero).

The bound on $\psi_i(t')$ implies that the solutions $\varphi_i(t_0+t')$ and $\varphi_i^p(\tau^+(t_0)+t')$, which coincide at one or more sites when $t'=0$, do not coincide anywhere for $t' > 0$. Since the local velocities can be shown to be finite,

$$\tau^+(t_0+t') < \tau^+(t_0) \text{ for } t' > 0 ; \quad (7)$$

that is, the bounding time $\tau^+(t)$ *decreases* or *improves* as t increases. Similarly, $\tau^-(t)$ is a monotonically increasing function, bounded above by $\tau^+(t)$. As the arbitrary solution at a given time and the corresponding bounding solutions always separate, *there is no fixed point for* $\Delta\tau$, *except at* $\Delta\tau = 0$. This implies that $\Delta\tau$ must approach 0 as $t \to \infty$. The arbitrary solution φ_i and the given positive velocity solution φ_i^p approach each other in the sense that there is a constant $\tau = \lim_{t\to\infty}\tau^+(t) = \lim_{t\to\infty}\tau^-(t)$ with

$$\lim_{t\to\infty} [\varphi_i(t) - \varphi_i^p(t+\tau)] = 0 \text{ for all } i . \quad (8)$$

The asymptotic equality Eq. (8) can be used to indirectly compare *any* two solutions, implying that all solutions approach a solution unique up to time translations, as $t \to \infty$.

Using uniqueness, it is easily seen that in CDW models, where V_i is periodic, the local velocities of the unique asymptotic solution are periodic in time; the shape of the configuration repeats itself with period $T = 2\pi/|v|$. Let $\varphi_i(t)$ be a solution to the equations of motion Eq. (1). The translation $\varphi_i(t) \to \varphi_i(t) + 2\pi$ gives another solution to the equations of motion. As $t \to \infty$, these solutions are related by a time translation, i.e.,

$$\varphi_i(t+T) - \varphi_i(t) - 2\pi \to 0 , \quad (9)$$

for some constant T, as $t \to \infty$, giving the stated periodicity.

I now summarize some applications to related models.

Continuous media.—These results can be extended to versions of Eq. (1) where the displacement is a continuously differentiable function of position $\mathbf{r}$, if the magnitude of the elastic forces increases with increasing $|\nabla\varphi|$ and the equations of motion are first order in time. An example of such a system, with strong nonlinearity, is given by the equation of motion [12],

$$\dot\varphi(\mathbf{r}) = h(\nabla\varphi)^2 + P(\nabla^2\varphi) + V'(\varphi,\mathbf{r}) + F(t) , \quad (10)$$

where $P(x)$ is a monotonically increasing function of x; such a model might be used to describe interface dynamics in a random medium, with φ being the interface height. The terms due to the drive force, the pinning force, and the gradient squared are identical at a point of tangential contact of two ordered configurations. The elastic term then separates configurations that are in contact, implying that the no-passing rule and uniqueness of the sliding state hold for this model (if the configuration widths are shown to be bounded).

SCF model.—The SCF model for CDW's is more complex than that of Eq. (1), as the pinning potential is not constant in the sliding CDW frame. By scaling space and time, the SCF model can be written as [4]

$$\dot\varphi = \nabla^2\varphi + U_z(\mathbf{r} + \varphi\hat{\mathbf{z}})\rho(\mathbf{r}) + F , \quad (11)$$

where $\hat{\mathbf{z}}$ is the direction of CDW modulation and transport, U is the impurity potential, and $\rho(\mathbf{r})$ is the charge density, which is periodic in $\hat{\mathbf{z}}$. In order for there to be a one-to-one mapping between the CDW coordinate $\mathbf{r}$ and the laboratory-frame position $\mathbf{R}$, given by $\mathbf{R} = \mathbf{r} + \varphi(\mathbf{r})\hat{\mathbf{z}}$, it is necessary to assume no phase slip [13,14] and $|\partial\varphi/\partial z| < 1$. Such constraints can be satisfied if the elastic constant increases as $\partial\varphi/\partial z$ increases (i.e., "hard" springs; for the case of simple spring forces, the gradient of φ is unbounded [14]). It is then easy to verify the no-passing rule and the uniqueness of the sliding state, given boundary conditions either periodic in the $\hat{\mathbf{z}}$ direction or such that no-passing explicitly holds at the boundaries of the sample in the laboratory frame (this is physically reasonable, as the formation of CDW's at the boundary will be affected by the interior phases). The periodicity of the sliding state is then the consequence of uniqueness and the invariance of the equations of motion under the combined operations $\varphi \to \varphi + 2\pi$, $\mathbf{r} \to \mathbf{r} - 2\pi\hat{\mathbf{z}}$. The local CDW velocities are periodic, measured *at a fixed point in* *the laboratory frame.*

Automata.—It is possible to write down, as approximations to CDW's in an ac field, synchronous automaton models, where time is discrete and the site variables are integers [15]. The proof of the uniqueness of the sliding state does not apply to the synchronous automaton or to full models of a CDW in an ac field, as the evolution of solutions can be compared only at a discrete set of times, rather than a continuous set. However, the no-passing rule and uniqueness of the average velocity *do hold* in these models.

Higher-dimensional order parameters.—The results shown here are *not* applicable to the case of a more complex parameter, where there is no natural partial ordering of the configurations, as a no-passing rule cannot be defined. For example, in defect-free flux lattices in type-II superconductors, distortions are described by a displacement vector which is two dimensional. It is easy to define *single-particle* models with two-component displacements which exhibit nonunique velocities. CDW models with phase slip, which can be described by nonconvex elastic potentials or a two-dimensional order parameter, are known to exhibit hysteretic v vs F behavior [13].

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[10] It is necessary to use bounds on configuration widths, defined as $(\max_i \varphi_i - \min_i \varphi_i)$, in a *finite* system, as shown in Ref. [7].

[11] Note that, given an arbitrary initial configuration, the rate of the relaxation of the velocity to its limiting value may depend on the system size.

[12] I would like to thank D. S. Fisher and T. Hwa for suggesting the consideration of such models.

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