# Dynamics and Memory Effects in Rupture of Thermal Fuse Networks

Didier Sornette and Christian Vanneste
*Laboratoire de Physique de la Matière Condensée, Université de Nice-Sophia Antipolis, Parc Valrose, 06034 Nice CEDEX, France*
(Received 30 September 1991)

A simple dynamical generalization of the electrical random fuse model for rupture in random media is introduced in which fuses are heated locally by a generalized Joule effect. When their temperature reaches a given threshold, the fuses burn out irreversibly and become insulators. In one limit, the rupture dynamics is spontaneously attracted to the critical state of the bond percolation model. In another limit, it recovers the "static" random fuse model previously studied in the literature. In between these two extremes, the existence of a novel dynamical memory effect produces a rich phenomenology of fractal rupture patterns, which are sensitively dependent upon the input current.

PACS numbers: 64.60.Ht, 05.40.+j, 62.20.Mk

Rupture of random materials within a statistical physics framework has been studied intensively these last few years and a partial classification of some different possible regimes has emerged [1,2]. Such an approach seems unavoidable in order to tackle the important difficulties underlying this field, stemming for instance from the presence of many interacting defects, from the long-range nature of electric or elastic Green functions, and from the irreversible evolution of the rupture. As a consequence of the extraordinary richness of the phenomenology of fracture mechanics, a large variety of lattice models have been defined, each model capturing to some degree a part of a realistic situation. However, all these models have one fundamental property in common, namely, that the evolution of the rupture is *quasistatic*. Indeed, the evolution is obtained by solving the elastic or electric equations and the first bond which fulfills the rupture criterion is singled out and broken. Then, the sample process is iterated and so on until a macroscopic fracture appears. In these models, there is no dynamics but only an irreversible process with no time scale. This is exactly the spirit of the growth models such as DLA (diffusion-limited aggregation), which describe the *quasistatic* irreversible evolution of complex interfaces [3].

However, natural materials very rarely break down without presenting time-dependent effects. The rupture of a window screen by a stone, more generally any breaking of colliding objects, cracking under explosions, rupture under fatigue, etc., are a few examples showing that the time evolution of a system is most important in order to describe its rupture adequately. How the dynamical aspects of the rupture processes control the nature and geometry of fracture is not understood in general. Time evolution implies that the different elements of a system might not find the time to relax to a steady state in a finite time. For instance, if a bond breaks down before complete relaxation, we should expect that the memory of previous deformations and stresses might bring novel effects on the evolution of the fracture process. Here, we present a new simple statistical model of rupture which encaptures some of these important dynamical and novel memory effects. For the sake of simplicity and demonstration of the concepts, we restrict the study to a scalar electrical rupture model. Inspired by the physics of a real fuse which burns out by melting, the model consists in a dynamical extension of the random fuse model [4] in which the temperature $T$ of a fuse of specific heat $C$, resistance $R$, and current $i$ obeys the equation

$$C\,dT/dt = Ri^b - aT\,, \tag{1}$$

where $Ri^b$ accounts for a generalized Joule heat source and $-aT$ describes the coupling to a thermal bath. The definition of the model is completed by the rule that a fuse burns out irreversibly when its temperature reaches the temperature threshold $T_r$ (chosen identical for all fuses). After such a breakdown, we assume that the current distribution in the remaining fuses adjusts itself instantaneously. The dynamics is thus solely controlled by the temperature evolution.

For random resistances with the same temperature threshold $T_r$ for rupture, we show that the *quasistatic* random fuse model [4] is recovered in the limit $b \to +\infty$, whereas the other extreme $b \to 0$ corresponds exactly to the percolation model. In the intermediate regime, the competition between the two time scales $T_r/Ri^b$ and $a^{-1}$ (taking $C=1$) produces a rich phenomenology of fractal rupture patterns, which present sensitive dependences upon the input current. In the limit of vanishing disorder or near the network rupture threshold, the cracks undergo branching instabilities and form dendritelike patterns.

Consider a 2D square lattice of unit mesh oriented at 45° with respect to the two borders a distance $L/\sqrt{2}$ apart. Periodic boundary conditions are assumed in the direction perpendicular to the two borders which act as bus bars. We assume that the conductances $G=R^{-1}$ of the bonds are distributed according to a given probability distribution $P_G(G)$, chosen in our computation to be uniform in the interval $[1-\Delta\sigma/2, 1+\Delta\sigma/2]$ with $\Delta\sigma$ varying from 0 to 2. At time $t=0$, a constant current $I$ is suddenly applied and flows from one bus bar to the other. The current per bond would thus be $I/L$ in the absence of resistance disorder. As in the quasistatic random fuse model, the electrical voltages and currents are assumed to have infinitely short response. We thus calculate the

 

current distribution in the network, i.e., the current $i_n$ in each bond, as for a static input current. We have used both a relaxation method and the conjugate gradient technique, using an error criterion $\varepsilon \leq 10^{-20}$. No significant difference was found between the two numerical schemes. Once the currents in each bond are known, it is reported in Eq. (1), which gives the time evolution of the $n$th fuse temperature $T_n(t)$ [see Eq. (2) below]. The first rupture occurs at $t=t_1$ on the bond which first reaches the temperature threshold $T_r$, chosen equal to unity for all bonds. After the first fracture, which amounts to putting its resistance equal to infinity, the currents in all remaining bonds are calculated again, assuming that they instantaneously reach their stationary values. This new set of currents $\{i_n(t_1)\}$ is injected back into the heat equation (1), with the additional information that the temperature attained by bond $n$ at $t_1$ is $T_n(t_1)$. Iterating the procedure again and again after each rupture and noting $t_i$, the time at which the $i$th rupture event occurs, and $\{i_n(t_i)\}$ and $\{T_n(t_i)\}$ the corresponding sets of bond currents and temperatures on all remaining bonds, we obtain the following general time-dependent temperature expression for the $n$th bond:

$$T_n(t)=T_n(t_i)e^{-a(t-t_i)} + \{R_n[i_n(t_i)]^b/a\}[1-\exp\{-a(t-t_i)\}], \quad (2)$$

for $t_i \leq t \leq t_{i+1}$ if $T_n(t) \leq 1$.

Note that the simplicity of the model stems from the separation of the time evolution of the electrical and thermal fields: The electric current distribution evolves instantaneously under the thermal rupture of a new bond and the thermal field changes continuously under the fixed current distribution until the next rupture occurs. This feature greatly simplifies the analysis and the numerical computations. Furthermore, this may be of some relevance to real material systems. Consider the rupture of metals for instance. There is an analogy between the thermal field in the random fuse network and the plastic component of the metal deformation field. The plastic deformation is indeed known to be preferentially localized in regions of large elastic deformation, for instance in the vicinity of crack tips. In particular, it is well known that the theoretical $r^{-1/2}$ divergence of the stress at the tip of a crack is smoothed out by the appearance of a plastic deformation localized near the crack tip. Thus, similarly to the extension and deformation of the plastic region, which is controlled by the value of the elastic stress, the thermal field is controlled by the electric field in our model. Both the plastic deformation and our temperature field present history-dependent properties. Furthermore, the growth of a crack involves first the restructuring of the plastic zone near the crack tip. Thus, the rupture is controlled by the plastic field, similarly to what occurs in our dynamical thermal model where the temperature controls the fracture and evolution of the network topology.

Before presenting our original results, it is worth pointing out that the thermal fuse model contains as natural limits two well-studied statistical models of rupture, namely, the *quasistatic* random fuse model [4] for $b \to +\infty$ and the bond percolation model [5] for $b \to 0$. When $b$ becomes very large, the heating rate $Ri^b$ of the bond which carries the largest current becomes much larger than those of all other bonds. In this limit, only the bond which carries the largest current is significantly heated compared to the others and it reaches the temperature threshold $T_r$ first. Since this remains true at all times, this limit $b \to +\infty$ corresponds to always burning out the bond which carries the largest current. Forgetting the time dependence of this process [6], which here is of no consequence for the geometry of the rupture, we thus recover exactly in the limit $b \to +\infty$ the *quasistatic* random fuse model [4]. In the other limit, $b \to 0$, the heating rates become independent of the current field and the thermal field only depends on the resistance distribution. The largest resistances will reach the rupture threshold $T_r$ after a finite time whereas the smallest ones will never burn out if their asymptotic temperature $R/a$ is smaller than $T_r$. Therefore, the successive bond breakdowns are independent random events solely controlled by the distribution and spatial position of the electrical resistances. When a continuous path of ruptured fuses appears, the rupture process stops since the global network resistance becomes infinite. At this point, the distribution and position of ruptured bonds is given exactly by the bond percolation model at its critical point $p=p_c$. The rupture dynamics is spontaneously attracted to the critical state of the bond percolation model [5], thus providing a new example of self-organized criticality [7]. The corresponding fraction $q=1-p$ of absent bonds at time $t$ is given by the fraction of the bonds which have attained the rupture threshold $T_r=1$, namely, $q=\int_1^{+\infty} P_T(T,t)dT$, where $P_T(T,t)$ is the distribution of the temperatures at time $t$ in the network. $P_T(T,t)$ is obtained by performing the change of variable $R \to T(R,t)$, where $T(R,t)$ is given by Eq. (2), in the initial resistance distribution $P_R(R)$:

$$P_T(T,t) = a[1-\exp(-at)]^{-1} P_R(a[1-\exp(-at)]^{-1}T).$$

Global rupture is reached when $q$ attains the percolation threshold $q_c=1-p_c$ ($=\frac{1}{2}$ for a square lattice in two dimensions).

Let us now study the generic case $0<b<+\infty$. Figure 1 presents typical examples of crack patterns at the end of the rupture sequence for the same system (with $b=2$ and $\Delta\sigma=0.2$) obtained by varying the applied current. $I_c$ is the minimum current under which the network presents a macroscopic fracture. Three regimes can be distinguished.

(i) For $I$ very close to $I_c$ [Fig. 1(a)], rupture is characterized by the growth of two or four main connected branches growing from a nucleating center, correspond-

ing to the first bond breaking in the network. The rupture threshold corresponds to the rupture of the resistance which presents initially the largest Joule heating power. As the quenched disorder decreases, the branches of the crack become straighter while presenting sidebranching instabilities. Decreasing the quenched disorder to very small values or allowing for a small annealed noise on a perfect network gives birth to cracks formed of four branches on which many secondary branches of random length have grown [6]. This corresponds to a selective amplification of noisy fluctuations near the advancing tip of the crack, similarly to a mechanism of sidebranching which has been proposed in dendritic solidification [8].

(ii) For $I \gg I_c$ [Fig. 1(c)], rupture occurs in two steps. At the beginning of the rupture process, the progressive deterioration is similar to a random bond dilution in which the initial quenched disorder on the resistances dominates the dynamics. At larger times, the dilution process is followed by a regime characterized by correlated cluster growth and fusion events between cracks. In this regime, current enhancement and screening effects become important as in case (i). The relative importance of these two steps depends upon the disorder $\Delta\sigma$ and on the exponent $b$. Increasing the disorder and decreasing $b$ favors the first uncorrelated random dilution regime. For a given resistance disorder configuration, it can be proven [6] that the ordered sequence of bond rupture and therefore the final crack pattern at the end of rupture do not change in the limit of large input currents $I$. This well-defined limit for the rupture process and final crack pattern for large $I$'s (see Fig. 2) is characterized by a dependence of the global rupture time scaling as $t_f \sim I^{-b}$. This limit amounts to neglecting the term $-aT$ in the right-hand side of Eq. (1) which couples the bonds to a thermal bath.

(iii) The intermediate case $I_c < I \ll +\infty$ shown in Fig. 1(b) corresponds to a crossover from the first regime $I \approx I_c$ to the last one $I \gg I_c$. It is characterized by a remarkable sensitivity with respect to the applied current. This can be guessed from Fig. 1(b) where, in addition to the main disconnecting crack, two large competing cracks bear witness to the existence of the previous rupture pattern which dominated in the first regime $I \approx I_c$ shown in Fig. 1(a). We have observed repetitively that very small variations of the applied current in the same system can lead to a catastrophic change in the breaking pattern. This can be tracked back to the existence of several large competing cracks. This property is analogous to the drastic alteration found on the best path in a random medium which undergoes slow drifts [9]. However, the drift con-

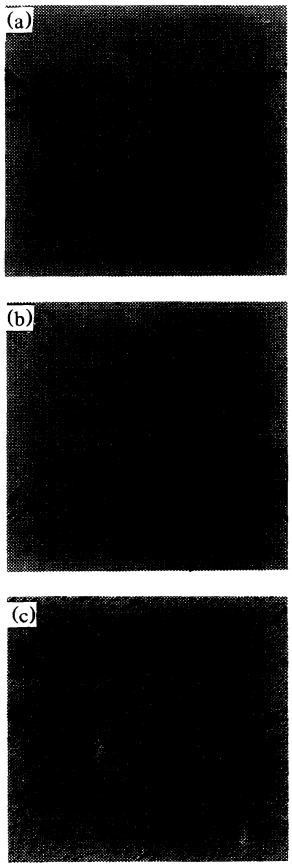


FIG. 1. Typical crack patterns at the final stage of rupture in a square lattice of size 180×180 tilted at 45°. The conductances are uniformly sampled in the interval [0.9,1.1]. The three pictures correspond to exactly the same disorder realization with, however, different applied currents: (a) Regime close to the rupture threshold, $I \approx I_c = 0.913$; (b) intermediate regime, $I = 1.000$; (c) asymptotic regime, $I = 30$ ($\gg I_c$).

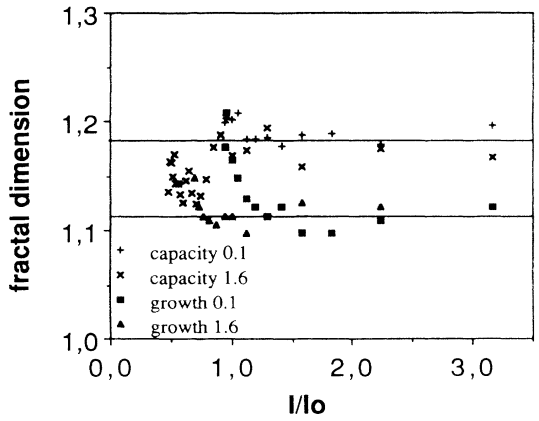


FIG. 2. "Growth" and "capacity" fractal dimensions (see text) in the case $b = 2$ as a function of the total current normalized to the threshold current $I_0$ for the uniform network ($\Delta\sigma = 0$). The data are presented for weak ($\Delta\sigma = 0.1$) and strong ($\Delta\sigma = 1.6$) disorder. Intermediate values of the disorder ($\Delta\sigma = 0.2$, $\Delta\sigma = 0.4$, $\Delta\sigma = 0.8$) exhibit identical behavior and provide the same values of $D_1$ and $D_2$. These results are obtained by averaging over 20 to 30 disorder configurations for each $I$ and $\Delta\sigma$, in systems of size $L = 80$. The two horizontal lines give the asymptotic values of the growth ($D_1 = 1.11 \pm 0.02$) and capacity ($D_2 = 1.18 \pm 0.02$) dimensions obtained for $I/I_0 \gg 1$.

cerns here the input current in the *same* random network.

In addition to macroscopic rupture pattern alterations, we have also observed a hierarchy of evolutions of the crack patterns at smaller scales, as the applied current $I$ is varied. Extending the proposed analogy with the problem of the best path in a random system, these observations suggest the existence of a hierarchical structure for the main rupture path, with global attractive basins of relatively wide extent in the $I$ variable, which break into smaller basins of attraction, corresponding to local readjustments of the path of rupture. Work is in progress to quantify these aspects. Note that these observations are in qualitative agreement with the recently conjectured correspondence [10] between the rupture pattern in the quasistatic random fuse model ($b \to +\infty$) and the random directed polymer problem. In our model, the existence of this wealth of behaviors and patterns relies on the competition between the two characteristic time scales $\tau_1 = T_r/Ri^b$ and $\tau_2 = a^{-1}$, in the presence of quenched disorder. When the heating time $\tau_1$ is the sole finite time constant ($a=0$), we recover the regime $I \gg I_c$ discussed above characterized by a complete lack of dependence of the rupture process with respect to the applied current. When the two time scales $\tau_1$ and $\tau_2$ are finite, a subtle additional *memory* effect appears whereby a bond with an intermediate heating power may break sooner than a bond which has first a lower heating rate and a higher heating rate later on. Indeed, from Eq. (1), the slope $dT_n/dt|_{t_i}$ at the beginning $t_i$ of the $(i+1)$th heating sequence of the $n$th resistance is $R_n[I_n(t_i)]^b - aT_n(t_i)$. Thus, the largest current does not lead necessarily to the largest heating efficiency if the temperature $T_n(t_i)$ is already high, since then a large heat flux is directed to the thermal bath. As a consequence, the bond which carries the instantaneous largest current is not generally the one chosen for rupture at a given time, in contrast to the rule of the quasistatic random fuse model [1,2,4].

Two different methods have been used for measuring the self-similar structure of the cracks. The first method computes a "growth" fractal dimension $D_1$ of the largest crack, at different times during the rupture, by calculating the total number of bonds it contains as a function of its projected length $l_x$ [respectively, $l_y$ and $(l_x l_y)^{1/2}$] along its principal growth direction (respectively, along the normal to this direction and their geometrical mean). The second method analyzes the macroscopic crack which is finally obtained at the end of the rupture, by measuring the number of broken bonds belonging to this crack in a box of size $l \times l$ as a function of $l$ ("capacity" fractal dimension $D_2$). For systems of size $L=80$, the capacity and "growth" fractal dimensions are estimated by a linear fit in a log-log plot over one and almost two decades, respectively. The results are displayed in Fig. 2 for $b=2$. The two methods give similar results with deviations which are, however, systematic, indicating, in a manner similar to DLA growth processes, that perimeter bonds on which the action is taking place are not characterized by exactly the same scaling as the "dead" bonds in the interior. For large currents, $D_1 = 1.11 \pm 0.02$ and $D_2 = 1.18 \pm 0.02$ (for $b=2$) do not depend on the disorder. As the current decreases and approaches $I_c$, $D_1$ increases up to the value 1.2 whereas $D_2$ exhibit variations in the range $[d_{\min}, 1.20]$ where $1.12 < d_{\min} < 1.18$ depends on the disorder. Calculations have also been performed for other $b$ values, whose results are obtained by averaging over 30 disorder configurations for each $I$ and $\Delta\sigma$, in systems of size $L=80$. For $b<2$, the fractal dimension increases ($D_{1,2} = 1.3 \pm 0.1$ for $b=1$, $D_{1,2} = 1.55 \pm 0.05$ for $b=0.5$), which culminates at $b=0$ where $D_2 = 1.88 \pm 0.02$ is very close to the percolation value $2-\beta/\nu = 273/144$. For $b>2$, the fractal dimension decreases down to $D_1 = D_2 \approx 1$ as found for $b=8$. This result is to be compared with the value for the quasistatic random fuse model [11]: $D = 1.1 \pm 0.1$.

A new dynamical model of statistical rupture in heterogeneous media has been introduced which incorporates a time dynamics with novel memory effects. A wealth of behaviors has been found which result from the competition and inter-relation between initial quenched disorder, current enhancement effects, and memory effects.

Laboratoire de Physique de la Matière Condensée is CNRS URA No. 190.

[1] See, e.g., *Statistical Models for the Fracture of Disordered Media,* edited by H. J. Herrmann and S. Roux (Elsevier, Amsterdam, 1990).

[2] *Disorder and Fracture,* edited by J. C. Charmet, S. Roux, and E. Guyon, NATO Advanced Study Institutes, Ser. B, Vol. 235 (Plenum, New York and London, 1990).

[3] See, e.g., *On Growth and Form,* edited by H. E. Stanley and N. Ostrowsky (Martinus Nijhoff, Kluwer Academic, Boston, 1986).

[4] L. de Arcangelis, S. Redner, and H. J. Herrmann, J. Phys. (Paris), Lett. **46**, L585 (1985).

[5] See, e.g., *Percolation, Structures and Processes,* edited by G. Deutscher, R. Zallen, and J. Adler, Annals of the Israel Physical Society Vol. 5 (Adam Hilger, Bristol, 1983).

[6] C. Vanneste and D. Sornette (to be published).

[7] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).

[8] R. Pieters and J. S. Langer, Phys. Rev. Lett. **56**, 1948 (1986); J. S. Langer, in *Chance and Matter,* Proceedings of the Les Houches Summer School, Session XLVI, 1986, edited by J. Souletie, J. Vannimenus, and R. Stora (Elsevier, Amsterdam, 1987).

[9] Y.-C. Zhang, Phys. Rev. Lett. **59**, 2125 (1987).

[10] A. Hansen, E. L. Hinrichsen, and S. Roux, Phys. Rev. Lett. **66**, 2476 (1991).

[11] L. de Arcangelis and H. J. Herrmann, Phys. Rev. B **39**, 2678 (1989).

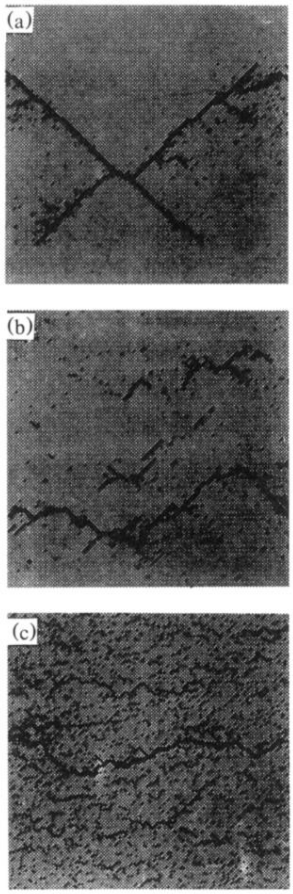


FIG. 1. Typical crack patterns at the final stage of rupture in a square lattice of size $180\times180$ tilted at 45°. The conductances are uniformly sampled in the interval [0.9,1.1]. The three pictures correspond to exactly the same disorder realization with, however, different applied currents: (a) Regime close to the rupture threshold, $I \approx I_c = 0.913$; (b) intermediate regime, $I = 1.000$; (c) asymptotic regime, $I = 30$ ($\gg I_c$).