Percolation and Fracture in Disordered Solids and Granular Media: Approach to a Fixed Point

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We argue that there exist universal fixed points (FPs) that classify the universality classes of fracture processes in disordered media. As a system approaches its macroscopic failure point, the ratio of its elastic moduli appears to approach a universal value, *independent of microscopic features of the system.* We suggest that there are *two* such FPs: one describes systems that are under a uniform external load and in which fracture does not take place at random, but depends on the stress field in the system, while the other describes systems in which fracture accumulates at random and is identical with the FPs of elastic percolation networks. Experimental data on fractured rocks appear to support this.

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Failure processes [1] play a fundamental role in many systems of industrial importance ranging from pressurized nuclear reactors and aircraft structures to the propagation of cracks in oil reservoirs and granular porous media, and in disordered solids such as alloys, ceramics, superconductors, and glasses. In the past few years, several simple models have been introduced for both electrical [2,3] and mechanical [4] failure of disordered media. These models are based on random networks in which each bond describes the system on a microscopic level, with failure characteristics described by a few control parameters. One applies an external potential, strain or stress, to the network and gradually increases it, as a result of which the individual bonds break irreversibly in a certain manner. Various properties of such failure processes have recently been investigated [5-8], and have been shown [8,9] to be capable of predicting several features of failure phenomena observed in experiments.

In this paper, we address four questions which we believe are of fundamental importance to the failure of disordered solids and granular porous media. (i) What is the *signature* (or a distinct property) of a system that is developing internal cracks and is close to its macroscopic failure point? If such a signature does exist, it may help one to detect and prevent the catastrophic failure of a system. On the other hand, in some systems we would like to make sure that macroscopic cracks *are* developed. For example, in order to increase oil production artificial fractures are created in oil reservoirs to increase the permeability of the system and, therefore, it is highly desirable to know whether such macroscopic fractures have actually been created. (ii) If there does exist a signature of a failing system, how *universal* is it? Does it depend on the microscopic properties of the system or the dynamics of failure and is material dependent? (iii) In general, the growth of cracks in a disordered system is a nonequilibrium and nonlinear phenomenon. On the other hand, static and linear properties of disordered systems are usually modeled by percolation networks of resistors or elastic bonds [10], in which the bonds are cut *at random.* Percolation phenomena represent second-order phase transitions, whereas at least some of the fracture phenomena modeled by the recent models [2-4] resemble first-order phase transitions. Under certain experimental conditions the accumulation of damage and the growth of cracks can be essentially at random as in, e.g., a system which is under rapid thermal cycling. In such a situation, a percolation process may be appropriate for describing the damage process. On the other hand, in the discrete models of mechanical breakdown [4-8], the failing system is under an external uniform *load* (e.g., stress or strain), and the breakdown of a bond does not take place at random, but depends on the stress or strain field around it. Therefore, it is important to know the extent of similarities between the properties of such networks and those of a percolation network. If there are any similarities between the two systems, then, percolation phenomena [10], which are now well understood and much easier to study, may help one to gain some insight about fracture of disordered systems under uniform external loads. (iv) How can one classify various fracture processes? One hopes that there are only a finite number of universality classes that contain most or all fracture processes and their universal properties.

To provide at least partial answers to the above questions, we have studied fracture phenomena using the fracture models introduced by Sahimi and Goddard [4]. We consider an $L\times L$ triangular network in which every site of the network is characterized by a displacement vector $\mathbf{u}_i$, and nearest-neighbor sites are connected by springs that can be stretched and bent. The case of a brittle material is studied here for which a linear approximation is valid up to a threshold defined below. The elastic energy of the system is given by

$$H=\frac{\alpha}{2}\sum_{\langle ij\rangle}e_{ij}[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{R}_{ij}]^2+\frac{\beta}{2}\sum_{\langle jik\rangle}e_{ij}e_{ik}(\delta\theta_{jik})^2, \qquad (1)$$

where $\mathbf{R}_{ij}$ is a unit vector from i to j, and e_{ij} the elastic constant of the bond between i and j (assumed to be unity). Here $\langle jik\rangle$ indicates that the sum is over all triplets in which the bonds ji and ik form an angle whose vertex is at i, α, and β denote the stretching and bond-bending force constants, respectively, and $\delta\theta_{jik}$ represents the change of angle between bonds ji and ik.

We now introduce a threshold value l_c for the length of a bond [4], which is selected according to the probability density function

$$f(l_c) = (1-\gamma) l_c^{-\gamma}, \tag{2}$$

where we use $\gamma = 0.80$ and 0. These two values of γ allow us to investigate the effect of the statistical distribution of l_c on the possible universality of the properties that we study. We use this power-law distribution because for $\gamma > \gamma_c$, where $\gamma_c = \frac{1}{4}$ is the critical value of γ, such distributions can give rise to unusual and nonuniversal phenomena [11,12]. We then initiate the failure process by applying an external strain on a fully connected network and determining the nodal displacements $\mathbf{u}_i$ by minimizing H with respect to $\mathbf{u}_i$ for all nodes i of the network. Two different methods have been used to model the failure process. In the first method, we select that spring for which the difference $l - l_c$ is maximum, where l is the current length of the spring in the strained network, and remove the spring from the system (break it). We call this model 1. In the second method, we remove *all* the bonds whose lengths have exceeded their thresholds. We refer to this as model 2.

After a spring (a set of springs) is broken, we recalculate the nodal displacements $\mathbf{u}_i$ for the new configuration of the network, select the next spring (set of springs) that is to be broken, and so on. If the external strain is not large enough to break any new spring, we gradually increase it. This process continues until the network fails macroscopically. We measure three properties of the network. We first distribute the threshold values l_c and measure the elastic modulus C_{11} of the network during the fracture process. We then use the *same* fully connected network (i.e., with the *same* distributed values of l_c), and measure the shear modulus μ of the network during the fracture process. This is equivalent to using two *identical* samples for measuring C_{11} and μ. During both measurements, we also monitor the force distribution (FD) of the network, i.e., the distribution of the forces that the unbroken springs of the network suffer [13]. The results presented below are for $L = 90$, for which we used a large number of realizations and averaged the results.

In Fig. 1 we present the ratio $r = C_{11}/\mu$, as a function of the fraction of unbroken springs, for various values of β/α, using model 1 of fracture. The last points on these curves represent C_{11}/μ right before the system fails macroscopically. We refer to this as the incipient fracture point (IFP). As can be seen, even though the initial states of the systems are different, they all approach the same value as the IFP is approached. Note that initially r remains essentially constant (which is similar to a percolation system, see below), i.e., it is *not* sensitive to a few cracks or even a collection of localized cracks. However, as damage accumulates and the cracks grow, a turning point (TP) appears and r changes drastically. Because $\beta/\alpha = 0$ corresponds to a system in which only central forces are present, Fig. 1 indicates that this behavior is independent of the microscopic force laws of the system. The behavior of the system for $\beta/\alpha = 1$ is particularly in-

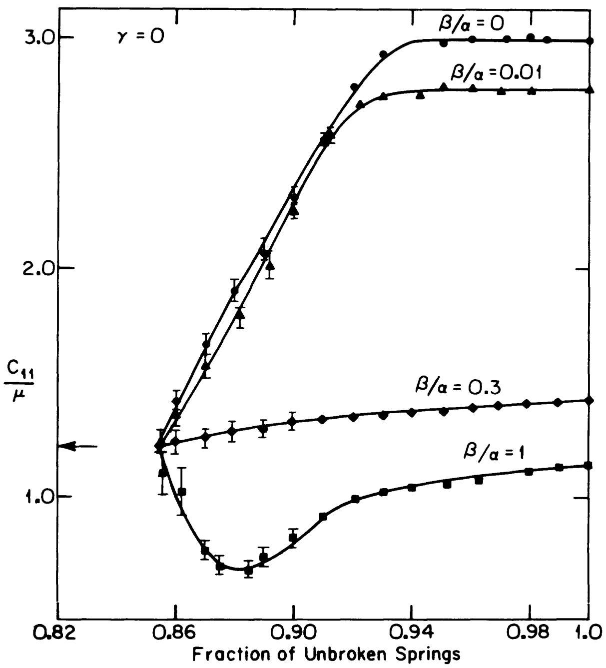

FIG. 1. The ratio C_{11}/μ vs the fraction of unbroken springs with model 1 of fracture.

teresting. Initially, r remains essentially constant. However, as damage accumulates a TP appears beyond which r decreases and reaches a minimum. But near the IFP, r rises again and approaches the value at the IFP which appears to be the same for all values of β/α.

To check whether this behavior depends on the dynamics of failure in systems that are under an external load, or the distribution of the threshold values, similar simulations with model 2 of fracture were carried out for $\gamma = 0$ and 0.8, the results of which are shown in Figs. 2 and 3. Although for $\gamma > \gamma_c$ many transport properties of percolation networks show anomalous behavior [11,12] near p_c, r appears to approach essentially the same value at the IFP as that for $\gamma = 0$. It is clear that, within the error bars, r appears to attain the same value in all cases as the IFP is approached, although the geometry of the macroscopic fracture is very different in models 1 and 2, particularly for $\gamma = 0$ and $\gamma = 0.8$. From Figs. 1-3, we may conclude that for *all* values of $\gamma < 1$ and β/α, and regardless of the dynamics of fracture, one has a *universal* fixed point

$$C_{11}/\mu \simeq 1.25. \tag{3}$$

The appearance of a universal fixed point may mean that in many disordered media in which fracture occurs the approach of r to the IFP and its universal value can be interpreted as the *signature* of a failing system. Although Figs. 1-3 indicate that for certain values of β/α one may have a nonmonotonic variation of r with the accumulated damage (which, from an experimental view, makes the closeness of r to its universal value useless as the signature of a failing system), for most real systems

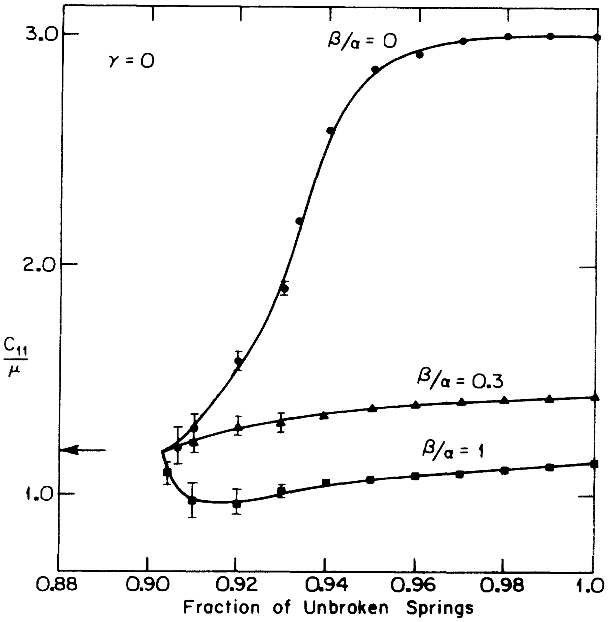

FIG. 2. The ratio C_{11}/μ vs the fraction of unbroken springs with model 2 of fracture.

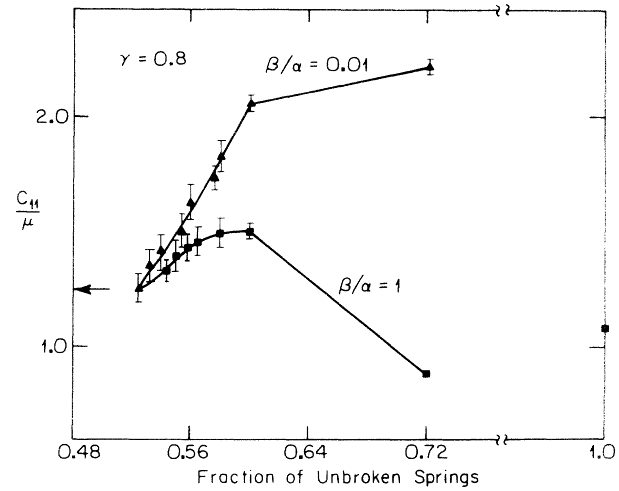

FIG. 3. The ratio C_{11}/μ vs the fraction of unbroken springs with model 2 of fracture.

one has [14] $\beta/\alpha \leq 0.3$, and for such values of β/α the approach of r to the IFP is *always* monotonic (see Figs. 1 and 2). This is also true for three-dimensional systems.

What is the theoretical explanation for this apparent universality of r? It is not difficult to show that both C_{11} and μ follow the same type of dependence on the fraction of unbroken springs as the IFP is approached. As such r represents an *amplitude ratio.* It is known [15] from statistical mechanics that certain amplitude ratios are *universal.* The apparent universality of r for the fracture problem indicates that one may be able to map this problem onto an equivalent statistical-mechanical system. Indeed, Blumberg Selinger *et al.* [16] have recently shown that one can develop a statistical-thermodynamic approach that associates fracture of a solid with the approach of a spinodal, or a metastable limit, upon increasing stress. The apparent universality of r may mean that, much like renormalization-group theory of critical phenomena, universal fixed points may be used for classifying various fracture processes. We now discuss this possibility.

To begin with, we note that it has been suggested that [17–19] for elastic percolation networks near p_c, C_{11}/μ approaches a universal value, *independent of the microscopic details of the system.* For two-dimensional isotropic systems near p_c one has [18] $C_{11}/\mu \simeq 3$. Although for fractured networks considered here, the value of C_{11}/μ at the IFP is different from that of percolation networks at p_c, the fact that in both systems C_{11}/μ approaches a universal value indicates that the two phenomena share some common features. Sahimi and Goddard [4] and Roux *et al.* [20] have argued that, in the limit $\gamma = 1$, i.e., the limit of *infinite* disorder, model 1 of fracture studied here is equivalent to a percolation process. However, in real materials disorder is finite and the fact that even for $\gamma = 0.8$ the value of r at IFP is very different from that of elastic percolation networks at p_c indicates that the limit $\gamma = 1$ may be a singular point, so that even for $\gamma = 1 - \epsilon$ ($\epsilon \ll 1$), one should still obtain the value of r at IFP found here and not that of percolation networks at p_c. However, there are more similarities between the two phenomena. In a typical brittle fracture process, the stress-strain diagram has a maximum, and we find that *up to the maximum point* in the diagram the FDs in the fractured and percolation networks are completely similar (details will be given elsewhere [21]). The reason is that at the initial stages of fracture of a disordered system, stress enhancement at the tip of the microcracks is not very large and microcracks can nucleate essentially at random. But because beyond the maximum the fractured system is in the so-called postfailure regime [8,22], in which it is highly sensitive to small variations in the external strain, microcracking no longer occurs at random, and the similarities between the two phenomena end. Indeed, the TPs in Figs. 1–3 correspond to the maximum in the stress-strain diagram of the fractured system. That the similarities between fracture and percolation processes end at the maximum in the stress-strain diagram is intuitively appealing and transparent.

The existence of a universal fixed point in a fractured system can be directly tested by experimental measurements. For example, for granular porous media, Schwartz, Johnson, and Feng [23] have proposed a model that can predict many experimental features of such systems. The percolation properties of this model have been shown [18,24] to be very similar to those of elastic percolation networks and, in particular, as p_c of the granular medium is neared, C_{11}/μ approaches a fixed point. Therefore, we expect a fracturing granular porous medium to show fixed-point behavior similar to what we find here. From an experimental point of view, this can be directly tested since

$$V_p/V_s = (C_{11}/\mu + \tfrac{1}{3})^{1/2}, \quad (4)$$

where V_p and V_s are the velocities of the shear and compressional waves in the medium, respectively. Sammonds *et al.* [25] fractured four samples of a sandstone at four different confining pressures. Different confining pressures result in different fracture patterns since they control the closure of preexisting cracks and the nucleation and growth of new microcracks. They also measured V_p and V_s during fracture. At the three lowest confining pressures the corresponding fracture patterns were found to be brittlelike, and from their results we find that $V_p/V_s \simeq 1.14 \pm 0.04$ at the IFP for all three fractured sandstones. At the highest confining pressure the fracture was ductilelike, and although the stress-strain diagram of such a system is *not* similar to that of brittle fracture, their results indicated that $V_p/V_s \simeq 1.1$, beyond the point at which stress became independent of strain (typical of ductile fracture). These data provide strong experimental support for the existence of universal fixed points at the IFP.

We thus propose that the value of r at the IFP can be used to classify various universality classes of fracture processes. Specifically, we propose that there are *two* distinct universality classes. One is for systems that are under a uniform external load (stress or strain) in which the growth of a crack at a point depends on the environment around that point and, therefore, the damage accumulation is *not* random. Such systems are described by the fixed point found here. The second one is for systems in which damage accumulates essentially at random. Such systems are described by the fixed point of elastic percolation networks at p_c. Note that, unlike the values of the critical exponents for the scaling of the external stress with the size of the system [7,8] which are currently controversial, the values of r for fracture and percolation processes are very distinct. We do, however, need to simulate three-dimensional systems to estimate C_{11}/μ at the IFP.

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