Continuum Damping of High-Mode-Number Toroidal Alfvén Waves

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An asymptotic theory is developed to determine the continuum damping of short-wavelength toroidal Alfvén eigenmodes, which is essential for ascertaining thresholds for alpha-particle-driven instability in ignited tokamaks.

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Magnetic fusion research has finally reached the point at which large-scale thermonuclear burning experiments can be undertaken. Thus it is essential to think about novel physics features that may arise from the presence of large numbers of fusion-product alpha particles whose speeds are greater than the Alfvén speed $v_A=B_0/(4\pi\rho)^{1/2}$. Some effects have already been seen from superthermal fast ions produced by neutral beams or rf heating [1,2]. It has been pointed out [3-6] that toroidal coupling of the alphas to the Alfvén wave might lead to instability and possible diffusive loss of the alpha particles. A principal uncertainty in estimating the critical threshold for the instability lies in calculating the damping rate of the Alfvén waves. In a sheared magnetic field these waves are highly localized at the surface $\omega=k_\parallel v_A$ and strongly damped in most cases. An exception occurs for the so-called toroidal Alfvén eigenmodes (TAE) [7] where, due to the periodic nature of the toroidal field, gaps can arise in the continuum frequency spectrum of localized modes, within which discrete modes (undamped in lowest order) exist and can be alpha-particle destabilized. Since the destabilization is weak, it is also necessary to calculate the damping with some precision. Within the ideal MHD equations one can describe damping independent of the detailed dissipation mechanism [8-10]. The purpose of this Letter is to present a detailed asymptotic linear theory for the damping of short-wavelength TAE modes in a large-aspect-ratio tokamak.

The continuum damping of high-n TAE modes has also been analytically studied recently [10] by means of the ballooning representation, in the low-shear limit. These results resemble ours, although the scaling with mode number reported in Ref. [10] is different.

We limit ourselves here to nearly circular equilibria and assume small inverse aspect ratio r/R, low β, and high average poloidal mode number m_0 (so $d/dr \gg 1/r$ while $m_0\varepsilon$ is finite).

The frequency of a linearized wave is determined from the stationarity of the Lagrangian:

$$\omega^2=\frac{\text{magnetic energy}}{\text{kinetic energy}}=\frac{\int d^3r[\nabla(\mathbf{b}\cdot\nabla\Phi)]^2}{4\pi\int d^3r\,\rho(\nabla\Phi)^2/B^2}. \quad (1)$$

In Eq. (1), Φ is the wave electrostatic potential, and $\mathbf{b}$ is a unit vector along the unperturbed field, $B\mathbf{b}=B_\varphi\hat{\varphi}+B_\theta\hat{\theta}$. Because of the equilibrium toroidal symmetry, we expand $\Phi=\exp[i(n\varphi-\omega t)]\sum_m\phi_m(r)e^{-im\theta}$ and, by varying Eq. (1) with respect to ϕ_m, arrive at the mode equations valid to first order in ε:

$$\frac{d}{dr}\left(\frac{\omega^2}{v_A^2}-k_{\parallel m}^2\right)\frac{d\phi_m}{dr}-\frac{m^2}{r^2}\left(\frac{\omega^2}{v_A^2}-k_{\parallel m}^2\right)\phi_m+\varepsilon\frac{\omega^2}{v_A^2}\left[\frac{d^2\phi_{m+1}}{dr^2}+\frac{d^2\phi_{m-1}}{dr^2}\right]=0. \quad (2)$$

In Eq. (2), $k_{\parallel m}=(1/R)[n-m/q(r)]$ with R and r the major and minor radii of the torus, and $q(r)=rB_\phi/RB_\theta$. The toroidal coupling factor is $\varepsilon=\sigma r/R\ll 1$, where the value of σ depends on the details of the equilibrium, e.g., $\sigma=\frac{5}{2}$ for a low-beta, Shafranov-shifted circular equilibrium [9]. In the cylindrical limit ($\varepsilon\to 0$), Eq. (2) has a singularity at the surface $\omega=k_{\parallel m}v_A$, which may be regularized by nonzero ε. Since the toroidal coupling is important only near the singularity, it is retained only in the highest-order derivatives of $\phi_{m\pm 1}$.

The essence of the TAE is that, for $\omega=v_A/2Rq$, modes m_0-1 and m_0 are both resonant at the point $r=r_0$, where $q(r_0)=(m_0-\frac{1}{2})/n$. We write $\omega^2=[v_A(r_0)/2Rq(r_0)]^2/(1-\varepsilon g_0)$, where g_0 represents the complex shift of the eigenfrequency, and introduce $m=m_0+l$, with $l\ll m_0$. Expanding Eq. (2) about the point r_l, where $q(r_l)=(m_0+l-\frac{1}{2})/n$ and where only the coupling of the l and $l-1$ harmonics is important, we obtain

$$\frac{d}{dx}\left[\frac{1+\varepsilon g_l}{4}-(x-l)^2\right]\frac{d\phi_l}{dx}-\frac{1}{s^2}\left[\frac{1}{4}-(x-l)^2\right]\phi_l+\frac{\varepsilon}{4}\frac{d^2\phi_{l-1}}{dx^2}=0, \quad (3)$$

$$\frac{d}{dx}\left[\frac{1+\varepsilon g_l}{4}-(x-l+1)^2\right]\frac{d\phi_{l-1}}{dx}-\frac{1}{s^2}\left[\frac{1}{4}-(x-l+1)^2\right]\phi_{l-1}+\frac{\varepsilon}{4}\frac{d^2\phi_l}{dx^2}=0, \quad (4)$$

where $x=nq(r)-m_0$, $s=d(\ln q)/d(\ln r)$, and $g_l\cong g_0+2l/m_0\hat{\varepsilon}$, with $\hat{\varepsilon}=\varepsilon[\partial\ln(q/v_A)/\partial\ln q]_{r=r_0}^{-1}$. The coupling of the l and $l+1$ harmonics near r_{l+1}, where $q(r_{l+1})=(m_0+l+\frac{1}{2})/n$, is also governed by Eqs. (3) and (4), with $l\to l+1$. Near a singular layer (say,

$x-l=-\frac{1}{2}$) where toroidal coupling is important, we find

$$\frac{d}{dy}\left[\left(\frac{\varepsilon g_l}{4}-y\right)\frac{d\phi_{l-1}}{dy}+\frac{\varepsilon}{4}\frac{d\phi_l}{dy}\right]=0\,,$$
$$\frac{d}{dy}\left[\frac{\varepsilon}{4}\frac{d\phi_{l-1}}{dy}+\left(\frac{\varepsilon g_l}{4}+y\right)\frac{d\phi_l}{dy}\right]=0\,, \qquad (5)$$

with $y=x-l+\frac{1}{2}$. For $\phi_l'\equiv d\phi_l/dy$ we obtain

$$\phi_l'=[yC_l-\tfrac{1}{4}\varepsilon(C_{l-1}+g_lC_l)]/[y^2+\tfrac{1}{16}\varepsilon^2(1-g_l^2)]\,, \qquad (6a)$$

$$\phi_{l-1}'=[yC_{l-1}+\tfrac{1}{4}\varepsilon(C_l+g_lC_{l-1})]/[y^2+\tfrac{1}{16}\varepsilon^2(1-g_l^2)]\,. \qquad (6b)$$

The integration constants C_l and C_{l-1} are the values of $y\phi_l'$ and $y\phi_{l-1}'$ away from the singular point. While $y\phi'$ is constant across the boundary, ϕ_l and ϕ_{l-1} are discontinuous across the boundary layer of width $\varepsilon/4$, with jumps given by $\Delta\phi_l\cong\int_{-\infty}^{\infty}\phi_l'dy=\pi(C_{l-1}\beta_l+C_l\alpha_l)$ and $\Delta\phi_{l-1}=-\pi(C_{l-1}\alpha_l+C_l\beta_l)$, with $\alpha_l=-g_l(1-g_l^2)^{-1/2}$ and $\beta_l=\alpha_l/g_l$. The branch of the square root is to be determined from the causality requirement that if $\omega_r>0$ all functions of g are analytic in the upper half plane. Hence along the real axis the square root is positive for $|g|<1$ and is $-i(g/|g|)(g^2-1)^{1/2}$ for $|g|>1$, manifesting damping. Branch cuts are taken from $g_l=\pm 1$ to $-i\infty$.

We must ensure that all harmonics ϕ_l vanish as $x_l=x-l\to\pm\infty$. Except near the singularities at $x_l=\pm\frac{1}{2}$ the harmonics obey Eq. (4) with $\varepsilon=0$. Thus, near the singularities, $\phi_l\sim(\ln|x_l^2-\frac{1}{4}|+\Delta)C_l$, where Δ takes on various values depending on boundary conditions. Three characteristic values of Δ are (a) $\Delta_\infty(s)$, the value at $x_l^2\gtrsim\frac{1}{4}$ for the solution well behaved as $|x_l|\to\infty$; (b) $\Delta_s(s)$, the value at $x_l^2\lesssim\frac{1}{4}$ for the solution that is symmetric about $x_l=0$, i.e., $C_l(\frac{1}{2})=C_l(-\frac{1}{2})$, where we denote $C_l(\pm\frac{1}{2})=C_l^{\pm}$; and (c) $\Delta=\Delta_a(s)$ for the antisymmetric solution, with $C_l^+=-C_l^-$.

We now trace the construction of ϕ_l. For $x_l\lesssim-\frac{1}{2}$, we must have $\phi_l=C_l^-(\ln|x_l^2-\frac{1}{4}|+\Delta_\infty)$. For $x_l\gtrsim-\frac{1}{2}$, the jump relationship gives

$$\phi_l=C_l^-(\ln|x_l^2-\tfrac{1}{4}|+\Delta_\infty+\pi\alpha_l+\pi\beta_lC_{l-1}^{\pm}/C_l^-)\,.$$

Since ϕ_l can also be expressed as a superposition of symmetric and antisymmetric solutions, we have

$$\phi_l=\lambda_lC_l^-(\ln|x_l^2-\tfrac{1}{4}|+\Delta_s)+(1-\lambda_l)C_l^-(\ln|x_l^2-\tfrac{1}{4}|+\Delta_a)$$

for $x_l\gtrsim-\frac{1}{2}$. Hence,

$$\Delta_\infty+\pi\alpha_l+\pi\beta_lC_{l-1}^{\pm}/C_l^-=\lambda_l\Delta_s+(1-\lambda_l)\Delta_a\,.$$

Further, for $x_l\lesssim\frac{1}{2}$, we have

$$\phi_l=C_l^+[\ln|\tfrac{1}{4}-x_l^2|+[\lambda_l\Delta_s-(1-\lambda_l)\Delta_a]/(2\lambda_l-1)]\,,$$

with $C_l^+=(2\lambda_l-1)C_l^-$. Finally we must ensure after the jump at $x_l=\frac{1}{2}$ that $\Delta=\Delta_\infty$; thus,

$$\frac{\lambda_l\Delta_s-(1-\lambda_l)\Delta_a}{2\lambda_l-1}-\pi\alpha_{l+1}-\pi\beta_{l+1}\frac{C_{l+1}^-}{C_l^+}=\Delta_\infty\,.$$

Solving for λ_l after eliminating the C^- coefficients, we obtain our basic recursion relationship between the C^+

TABLE I. Values of the $\bar{\Delta}$ and $\tilde{\Delta}$ parameters, the normalized complex frequency $g_0^{(0)}$ of Eq. (8), and the G and $H_\pm$ functions in Eqs. (14) and (15), as functions of the shear s.

s	$\bar{\Delta}$	$\tilde{\Delta}$	Re $g_0^{(0)}$	Im $g_0^{(0)}$	G	H_+	H_-
small s	$\pi s/4$	$-\exp(-1/s)$	$-1+\pi^2s^2/8$	$-(16/\pi s)\exp(-2/s)$	$(s/2)^{1/2}\pi^{-3/2}$	$2/s$	$\pi^2 s/8$
0.3	.2605	-.0621	-.8740	-.00329	.1366	4.515	.09512
0.4	.3831	-.1669	-.7553	-.02797	.1810	2.467	.07606
0.5	.5392	-.3206	-.6065	-.10238	.2337	1.270	.05844
0.6	.7326	-.5190	-.4620	-.2320	.2967	.6250	.04472
0.7	.9650	-.7594	-.3438	-.3968	.3712	.3072	.03455
0.8	1.2369	-1.0405	-.2532	-.5761	.4585	.1560	.02705
0.9	1.5483	-1.3613	-.1835	-.7586	.5594	.08320	.02150
1.0	1.8993	-1.7213	-.1282	-.9404	.6747	.04667	.01733
1.2	2.7193	-2.5578	-.0445	-1.298	.9520	.01686	.01169
1.4	3.6959	-3.5486	.0175	-1.648	1.2970	7.09×10^{-3}	8.24×10^{-3}
1.6	4.8281	-4.6929	.0662	-1.993	1.7161	3.35×10^{-3}	6.02×10^{-3}
1.8	6.1153	-5.9905	.1059	-2.334	2.2154	1.73×10^{-3}	4.53×10^{-3}
2.0	7.5570	-7.4413	.1389	-2.671	2.8010	9.58×10^{-4}	3.49×10^{-3}
2.2	9.1529	-9.0449	.1669	-3.007	3.4787	5.63×10^{-4}	2.75×10^{-3}
2.4	10.9027	-10.8016	.1910	-3.340	4.2544	3.47×10^{-4}	2.20×10^{-3}
2.6	12.8062	-12.7111	.2118	-3.671	5.1336	2.22×10^{-4}	1.79×10^{-3}
2.8	14.8632	-14.7735	.2301	-4.001	6.1220	1.47×10^{-4}	1.48×10^{-3}
3.0	17.0736	-16.9887	.2462	-4.330	7.2250	1.00×10^{-4}	1.23×10^{-3}
large s	$6s^2/\pi$	$-\bar{\Delta}+\frac{1}{\pi s}$	1	$-s\pi/2$	$\frac{\sqrt{2}}{4\pi^{3/2}}3^{5/4}s^{11/4}$	$\pi^2/(15\cdot3^{5/2}s^{11/2})$	$\frac{1}{2\pi^2 s^3}$

coefficients (henceforth we drop the superscript),

$$C_l\left[\frac{\beta_{l+1}^2-(\alpha_{l+1}+\bar{\Delta})^2}{\bar{\Delta}+\alpha_{l+1}}+\frac{\tilde{\Delta}^2}{\bar{\Delta}+\alpha_l}\right]=C_{l-1}\left[\frac{\beta_l\tilde{\Delta}}{\bar{\Delta}+\alpha_l}\right]+C_{l+1}\left[\frac{\beta_{l+1}\tilde{\Delta}}{\bar{\Delta}+\alpha_{l+1}}\right]. \quad (7)$$

We have defined $\bar{\Delta}(s)=(1/2\pi)[2\Delta_\infty-(\Delta_s+\Delta_a)]$ and $\tilde{\Delta}(s)=(\Delta_s-\Delta_a)/2\pi$, and their values are shown in Table I. Note that $\tilde{\Delta}<0$, $\bar{\Delta}>0$, and $\bar{\Delta}>|\tilde{\Delta}|$. The eigenvalue g_0 must be determined by the requirement that a solution to Eq. (7) can be found for which $C_l\to 0$ as $|l|\to\infty$. Note that if g_0 is an eigenvalue, then $g_0+2j/m_0\hat{\varepsilon}$ with j any integer will also be an eigenvalue. Our principal interest is thus in the imaginary part of g_0.

For given s and $m_0\hat{\varepsilon}$, Eq. (7) may be solved numerically. It can also be solved analytically in two asymptotic limits. First consider the case $m_0\hat{\varepsilon}\ll 1$. Then all g_l except g_0 will be large and for $l\neq 0$, $\alpha_l=-i$, $\beta_l=0$. Then Eq. (7) is solved with only C_0 and C_{-1} nonzero, yielding

$$g_0^{(0)}=\lim_{m_0\hat{\varepsilon}\to 0} g_0=-\frac{1-y^2}{1+y^2}, \quad (8)$$

with $y=\bar{\Delta}-\tilde{\Delta}^2/(\bar{\Delta}-i)$. Since $\bar{\Delta}>0$, Eq. (8) gives $\mathrm{Im}g_0^{(0)}<0$, implying damping. Values of $\mathrm{Re}g_0^{(0)}$ and $\mathrm{Im}g_0^{(0)}$ are shown in Table I and provide an upper limit to the damping expected at finite $m_0\hat{\varepsilon}$.

We turn now to the interesting asymptotic case [10] $m_0\hat{\varepsilon}\gg 1$. In this limit the coefficients in Eq. (7) vary only slightly with l and we employ a finite-difference variant of WKB theory. We write Eq. (7) in the form $C_l(X_{l+1}+Y_l)=C_{l-1}W_l+C_{l+1}W_{l+1}$, where the subscripts on X, Y, and W indicate the dependence on l arising from g_l and hence α_l and β_l. Letting

$$C_l=\prod^l Q_j=\exp\sum^l \ln(Q_j)\approx\exp\left[\int^{l+1/2}dl'\ln Q(l')\right] \quad (9)$$

and then expanding in orders of $g'\equiv dg/dl=2/m_0\hat{\varepsilon}$ with $Q=Q_0+Q_1g'$, we have

$$Q_0(l)+\frac{1}{Q_0(l)}=\frac{X_l+Y_l}{W_l}\equiv 2f_l\,,$$
$$\frac{Q_1}{Q_0}=\frac{X'-Q_0W'-Q_0'W}{W(Q_0-1/Q_0)}\,, \quad (10)$$

with the primes denoting differentiation with respect to g. Equation (10) is solved to find $Q_0=f-(f^2-1)^{1/2}$ with

$$f=[2g_l\bar{\Delta}+(1-\bar{\Delta}^2+\tilde{\Delta}^2)(1-g_l^2)^{1/2}]/2|\tilde{\Delta}|\,.$$

For $|f|<1$, we have $|Q_0|=1$, representing an oscillating behavior of the C_l, whereas for $|f|>1$, as occurs for $|g|\sim 1$ (since $\bar{\Delta}>|\tilde{\Delta}|$), the solution that decays as $|l|\to\infty$ is chosen. Then Q_1 can be found from Eq. (10) and the integral $\int(Q_1/Q_0)dg$ performed to yield

$$C_{l-1}=\frac{\Gamma}{(f^2-1)^{1/4}}\frac{[g_l-\bar{\Delta}(1-g_l^2)^{1/2}]\exp\{\int^{g_l}\ln[f-(f^2-1)^{1/2}]dg/g'\}}{\{1+\tilde{\Delta}(1-g_l^2)^{1/2}[f-(f^2-1)^{1/2}]\}^{1/2}}\,, \quad (11)$$

with Γ a normalizing constant. In the oscillating region we have $|C_{l-1}|\sim\Gamma(1-f^2)^{1/4}|-g_l+\bar{\Delta}(1-g_l^2)^{1/2}|^{1/2}$, and the usual WKB joining condition applies.

To explain the global structure of the TAE mode, we show in Fig. 1 a schematic of the toroidal shear-Alfvén continuum resonance curves, $g_0^{\mathrm{res}}(x,l)=-(2l/m_0\hat{\varepsilon})\pm[1+16(x-l-\frac{1}{2})^2/\varepsilon^2]^{1/2}$, as a function of radial position $x=n(q-q_0)$, for a succession of poloidal harmonic numbers $m=m_0+l$, where $q_0=m_0/n$. The harmonic $\phi_l(x)$ will have a dissipative response where continuum resonance, $g_0=g_0^{\mathrm{res}}$, occurs. However, in the region where $g_l^2<1$, with $g_l=g_0+2l/m_0\hat{\varepsilon}$, no such resonance occurs. Each individual harmonic has only a limited radial extent, being localized where $|x-l|<\frac{1}{2}$. However, toroidal coupling permits the broad excitation of a global-type mode. In the region where $|f(l)|<1$, the adjacent harmonics have nearly equal amplitudes and shifted phase; i.e., for $C_l\propto\cos(\psi)$, $C_{l\pm 1}\propto\cos(\psi\pm\theta_l)$, with $\cos\theta_l=f(l)$ and ψ the phase. This "wavelike" pattern exists in the region $(m_0\hat{\varepsilon}/2)(g_--g_0)<x<(m_0\hat{\varepsilon}/2)(g_+-g_0)$, where $f(g_\pm)=\pm 1$. Outside of this region, the functions C_l are evanescent, with $C_{l+1}/C_l=\exp(\mp|\tilde{\theta}_l|)$, where the $\mp$ signs correspond to $g>g_+$ and $g<g_-$, respectively, and $\cosh\tilde{\theta}_l=|f(l)|$. Since $|g_\pm|<1$, the dissipation due to the continuum resonances at $g^2=1$ occurs where the mode amplitudes have exponentially decreased from their level in the wavelike region. Therefore, the

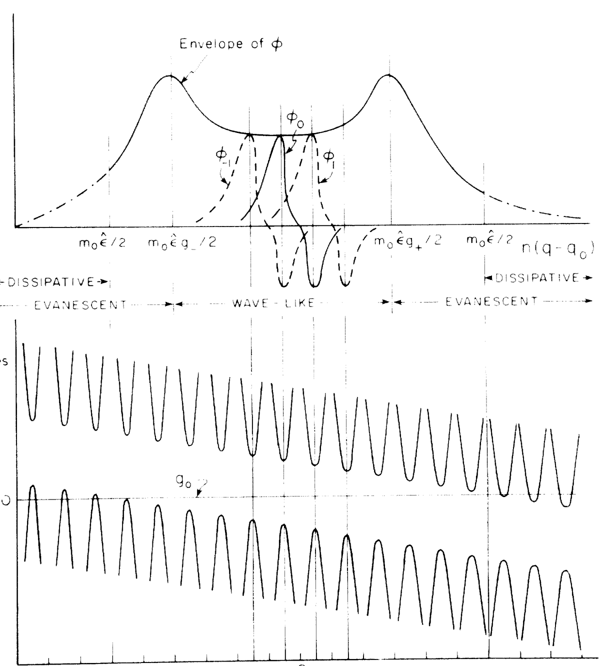

FIG. 1. Schematic plots of the toroidal shear-Alfvén continuum resonance curves $g_0^{\mathrm{res}}(x,l)$ and of several TAE harmonics $\phi_l(x)$ and their global envelope, as functions of radial position $x=n(q-q_0)$ with $q_0=m_0/n$.

damping decrement should be proportional to the tunneling factors $\exp[-2\int_{g_\pm}^{1}(dg/g')\tilde{\theta}_l]$.

In order to calculate the damping rate we construct a quadratic form from Eq. (7):

$$I=\sum_l |C_l|^2\left\{\frac{1+\bar{\Delta}^2}{\bar{\Delta}+\alpha_{l+1}}-2\bar{\Delta}\right\}+\sum_l |C_l|^2\frac{\tilde{\Delta}^2}{\bar{\Delta}+\alpha_l}-\sum_l (C_l^* C_{l+1}+C_{l+1}^* C_l)\frac{\beta_{l+1}\tilde{\Delta}}{\bar{\Delta}+\alpha_{l+1}}=0. \quad (12)$$

For real g, I will have an imaginary part arising from α and β for $|g|>1$. Moreover, for large $m_0\hat{\varepsilon}$ where C_l decays rapidly as $|g|$ increases, the main contribution to ImI will come from $|g|$ close to 1. The damping rate will be determined from the relation $(\mathrm{Im}g)(dI/dg)+\mathrm{Im}I=0$. Using Eqs. (10)-(12), we finally obtain an analytic expression for the damping rate $\gamma=\mathrm{Im}\omega$ when $m_0\hat{\varepsilon}\gg 1$:

$$\mathrm{Im}g^{\mathrm{as}}=2\gamma/\varepsilon\omega_0=-\frac{G(s)}{(m_0|\hat{\varepsilon}|)^{3/2}}[\exp(-m_0|\hat{\varepsilon}|H_+)+\exp(-m_0|\hat{\varepsilon}|H_-)], \quad (13)$$

with

$$G(s)=\frac{\sqrt{2}}{8\sqrt{\pi}}\frac{1+\bar{\Delta}^2-\tilde{\Delta}^2}{(\bar{\Delta}^2-\tilde{\Delta}^2)^{1/2}}\left\{\ln\left[\left(\frac{\bar{\Delta}^2}{\tilde{\Delta}^2}-1\right)^{1/2}+\frac{\bar{\Delta}}{|\tilde{\Delta}|}\right]\right\}^{-3/2} \quad (14)$$

and

$$H_\pm(s)=\cosh^{-1}B-\frac{B(B^2-1)^{1/2}}{A^2+B^2}+\frac{|A|}{A^2+B^2}[F(k,\phi)-E(k,\phi)]\mp\frac{A}{(A^2+B^2)^{1/2}}[K(k)-E(k)]. \quad (15)$$

Here $A=(1+\tilde{\Delta}^2-\bar{\Delta}^2)/2\tilde{\Delta}$, $B=-\bar{\Delta}/\tilde{\Delta}$, $k^2=(A^2+B^2-1)/(A^2+B^2)$, $\phi=\sin^{-1}[|A|/(A^2+B^2-1)^{1/2}]$, and $F(k,\phi)$ and $E(k,\phi)$ are the usual elliptic integrals, with $K(k)=F(k,\pi/2)$ and $E(k)=E(k,\pi/2)$. Values of $G(s)$ and $H_\pm(s)$ are shown in Table I.

An interpolation of Eqs. (8) and (13) is $(\mathrm{Im}g)^{-1}=(\mathrm{Im}g_0^{(0)})^{-1}+(\mathrm{Im}g^{\mathrm{as}})^{-1}$. Comparison with the numerical solution of Eq. (7) shows the following features:

(1) The analytic values asymptote satisfactorily to the numerical results at large $m_0\hat{\varepsilon}$. Although the large-$m_0\hat{\varepsilon}$ analysis assumes a large exponentially decaying region, it seems to remain valid even for moderate and small $m_0\hat{\varepsilon}$, where typically the analytic damping rate may be 30% higher than the numerical values.

(2) The numerical results show considerable oscillation in the damping rate as $m_0\hat{\varepsilon}$ is varied, which may be expected from the discrete nature of the sum for ImI from Eq. (12). At smaller s and $m_0\hat{\varepsilon}$ these oscillations are enhanced because in the numerics the sign of $(g^2-1)^{1/2}$ is discontinuously changed when crossing the branch cuts. To study whether interesting contributions to wave evolution arise from the branch cuts, it is necessary to resolve them by introducing finite Larmor radius and resistive effects, a subject to be considered in a subsequent paper.

Although our results are not valid if the overall eigenmode extends beyond the region where the linear expansion for $q^2(r)/v_A^2(r)$ is satisfied, the WKB procedure is easily generalized to treat this case.

Finally, we may draw three general conclusions. Damping is very weak at small shear ($s<0.5$), which makes instability likely in the center of the discharge. (Estimates of the TAE growth rate due to fast ion resonance are given, e.g., in Refs. [5] and [9].) Next, damping decreases strongly with m_0 (and hence n); however, the alpha-drive dependence with m can be diminished due to finite-Larmor-radius and banana effects [11], while other nonideal damping, e.g., due to the parallel electric field, will increase with m. Thus intermediate values such as those seen in recent experimental observations of the TAE instability [1,2] are most dangerous. Finally, profiles with large $\hat{\varepsilon}=\varepsilon[\partial\ln(q/v_A)/\partial\ln q]^{-1}$ should be susceptible to alpha-particle-driven instability and alpha-particle loss since the mode extends for large radial distances and the damping is small.

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