Knot Invariants as Nondegenerate Quantum Geometries

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The loop-space representation based on Ashtekar's new variables has allowed for the first time the construction of quantum states of the gravitational field. However, all states known up to the present were associated with spacetime metrics that were everywhere degenerate. In this Letter we present a new exact solution of the constraint equations of quantum gravity that is the first quantum state of the gravitational field known to be associated with a not-everywhere-degenerate metric. The state is associated with the second coefficient of the Alexander-Conway polynomial of knot theory.

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It is nowadays widely accepted that the road to a quantum theory of the gravitational field must be nonperturbative. Among the several nonperturbative proposals to quantize the gravitational field, the Dirac canonical quantization of Einstein's general relativity has received a great deal of attention since the introduction of a new set of canonical variables by Ashtekar [1] gave a tractable structure to the quantum constraint equations. In the Dirac procedure, after choosing a suitable set of classical canonical variables and a quantum polarization to represent the Poisson bracket algebra of the variables, the next step is to construct the set of physical states which are invariant under the symmetries of the theory considered. Only after the construction of this set is accomplished, by searching for those states annihilated by the constraint equations, is an inner product imposed on the physical states and physical predictions made. Up to now, our understanding of quantum gravity has only allowed us to know a small degenerate sector of the set of physical states [2-4]. The intention of this Letter is to present the first of a possible series of not-everywhere-degenerate quantum states of the gravitational field.

The new variables introduced by Ashtekar [1] consist of a densitized triad $\tilde{E}_i^a$ and a (complex) SU(2) connection A_a^i as conjugate momenta. These variables naturally imbed the phase space of general relativity in that of a Yang-Mills SU(2) theory. In particular, they allow us to understand the dynamics of general relativity in terms of a connection rather than a metric. This in turn brought to the forefront the role of Wilson loops in quantum gravity. Jacobson and Smolin [2] showed that in the connection representation of quantum gravity based on Ashtekar's new variables the trace of the holonomy of the Ashtekar connection along a smooth loop was a solution of the Hamiltonian constraint of quantum gravity (Wheeler-De Witt equation). Rovelli and Smolin [5] later introduced a whole new representation of quantum gravity where wave functions depend on loops: the loop space representation. Such a representation is known to exist for several other gauge theories, such as Maxwell's electromagnetism [6,7], Yang-Mills theories in the continuum [8] and lattice [9-11], Chern-Simons theories [12], linearized gravity [13], general relativity in 2+1 dimensions [14], and topological field theories [15]. The solutions found by Jacobson and Smolin in the connection representation found a natural counterpart in the loop space representation of quantum gravity—with the advantage that in this case they satisfied *all* the constraint equations—by considering wave functions that had support on diffeomorphism-invariant classes of loops (link classes).

It was noticed, however, that if one considered the quantum metric operator acting on these wave functions, the result was a degenerate metric. Not only did it have support distributionally along the loop, but its determinant was identically zero everywhere. It was hoped that a cure to this problem could come through the consideration of intersecting loops, since the degeneracy of the metric operator was lower at the points where there existed multiply defined tangents to the loop. However, analysis of two [2], three [3], and finally N [4] intersecting loop solutions in the connection representation showed that for an arbitrary finite number of intersecting loops the metric continued to be degenerate. The importance of the issue of the degeneracy of the metric appears more clearly when one considers general relativity with a cosmological constant. The only change in the Hamiltonian theory that a cosmological constant introduces is that a term proportional to the square root of the determinant of the metric is added to the Hamiltonian of the vacuum theory. It is therefore evident that if a quantum state is annihilated by the determinant of the metric and the Hamiltonian of the vacuum theory, it is also a quantum state for an arbitrary value of the cosmological constant. Since we know that general relativity, at least classically, has a completely different behavior depending on the value of the cosmological constant, it is quite clear that quantum states should also differ for different values of the cosmological constant. This shows that the states found up to the present are just a small and degenerate sector of the full space of states of the theory.

The intention of this Letter is to present the first solution to the constraint equations of quantum general relativity which is nondegenerate in the sense indicated above (at least in one point of the manifold). The main novelty consists in working *directly in the loop representation*, thus avoiding the no-go result of Ref. [4] that shows that all known solutions in the connection representation are degenerate. In order to do this we will rely on a series of techniques that allow us to write the constraints of quantum gravity and the wave functions directly in the loop representation.

The wave function we present here can be written as

$$\Psi[\gamma] = \rho_1(\gamma) + \rho_2(\gamma) , \quad (1)$$

$$\rho_1(\gamma) = h_{ax\,by\,cz} X^{ax\,by\,cz}(\gamma) , \quad (2)$$

$$\rho_2(\gamma) = g_{ax\,cz} g_{by\,dw} X^{ax\,by\,cz\,dw}(\gamma) , \quad (3)$$

where $g_{ax\,cz} = \epsilon_{acb}(x-z)^b/|x-z|^3$ is the propagator of a Chern-Simons theory, $h_{ax\,by\,cz} = \int d^3w\, g_{ax\,dw} g_{by\,ew} g_{cz\,fw} \times \epsilon^{def}$, and γ is a loop. We have introduced a "generalized Einstein convention" for the spatial indices meaning $g_{ax\cdots} X^{ax\cdots} = \int d^3x\, g_{ax\cdots} X^{ax\cdots}$. The X's are "coordinates" on loop space defined by

$$X^{a_1x_1 \cdots a_nx_n}(\gamma) = \oint_0^1 ds_1 \cdots \oint_0^1 ds_n\, \dot\gamma^{a_1}(s_1) \cdots \dot\gamma^{a_n}(s_n) \delta^3(x_1 - \gamma(s_1)) \cdots \delta^3(x_n - \gamma(s_n)) \Theta(s_1, \ldots, s_n) , \quad (4)$$

where $\dot\gamma^a(s)$ is the tangent to the loop γ at the point s and $\Theta(s_1, \ldots, s_n) = 1$ if $s_1 < \cdots < s_n$ and zero otherwise. These "coordinates" on loop space were introduced by Gambini and Leal [16] but appear implicitly in the earlier work of Makeenko and Migdal [17]. The expression for $\Psi[\gamma]$ has appeared in the literature in connection with the perturbative treatment of Chern-Simons theories [18]. In this context it was recognized as a knot invariant closely related to the second coefficient of the Alexander-Conway knot polynomial [$a_2 = \frac{1}{12}(\rho_1 + \rho_2 + \frac{1}{2})$ in an R^3 topology] and also with the Arf and Casson knot invariants [18]. The reader may feel uneasy with the fact that expressions that involve a background metric and particular choices of coordinates can be knot invariants. We will offer explicit proof that the wave functions are annihilated by the diffeomorphism constraint. Intuitive feeling can be gained by considering the knot as carrying electric currents and analyzing the meaning of the expressions from an electromagnetic point of view. These kinds of reasonings were known to Maxwell himself [19], and gave the initial impetus to knot theory.

The main result of this paper is to note that the wave function $\Psi[\gamma]$, when evaluated on knots with at most a triple self-intersection (the case of more complicated intersections will be discussed in [20]), is annihilated by the Hamiltonian constraint of quantum gravity and is not annihilated by the determinant of the metric (at the point of self-intersection). Starting from this solution we can construct a proper wave function on the complete loop space by the following prescription: (a) Let it be equal to $\Psi[\gamma]$ for single loops with self-intersections of order less than or equal to three. (b) Using the SU(2) spinor identity (see below) the value of the wave function can be specified on multiloops of up to three intersecting loops. (c) For higher multiloops or for loops with self-intersections of order higher than three, the wave function is defined to be zero (this does not imply the appearance of any "distributional" terms when evaluating the constraints since they do not connect the case of four or higher order with the lower ones).

This wave function is the first not-everywhere-degenerate solution to all the constraints of quantum gravity that is known. It can also be checked explicitly that it is annihilated by the diffeomorphism constraint, confirming its knot-invariant nature. This solution could not be seen in the previous attempts to construct N intersecting loop solutions in the connection representation [4], since being diffeomorphism invariant it would require an infinite superposition of holonomies in the connection representation to express it.

We now show that our wave function satisfies the conditions required to become a nondegenerate quantum state of the gravitational field. First of all, in the loop space representation, any wave function must satisfy $\Psi[\gamma_1^{-1}, \gamma_2] = \Psi[\gamma_1, \gamma_2]$, where γ_1^{-1} is a loop component of the multiloop with the opposite orientation; also $\Psi[\gamma_1, \gamma_2] = \Psi[\gamma_1\gamma_2] + \Psi[\gamma_1\gamma_2^{-1}]$ [SU(2) spinor identity], which states that the wave function for a multiloop composed of an arbitrary number of loops can be determined by its value on a single loop. All these conditions are met by the wave function presented, by construction. Second, to become states of the gravitational field, wave functions have to be annihilated by the diffeomorphism and Hamiltonian constraints. These constraints take on the following form when written in the loop representation [21,22]:

$$\hat C(v)\Psi[\gamma] = \int ds\, v^a(\gamma(s))\dot\gamma^b(s)\Delta_{ab}(s)\Psi[\gamma] , \quad (5)$$

$$\hat H(N)\Psi[\gamma] = \int ds \int dt\, N(\gamma(s)) f_\epsilon(\gamma(s), \gamma(t)) \times \dot\gamma^a(s)\dot\gamma^b(t)\Delta_{ab}^{(1)}\Psi[\gamma_s^t, \gamma_t^s] , \quad (6)$$

where Δ_{ab} is the area derivative, and f_ϵ is a regulator such that the constraint is retrieved in the limit $\epsilon \to 0$ where $f_\epsilon(y,z) \to \delta^3(y,z)$. γ_s^t denotes the portion of the loop going from s to t. The symbol $\Delta_{ab}^{(1)}$ acting on a wave function depending on a multiloop denotes an area derivative with respect to the first loop in the multiloop (separated by commas in our notation). We have integrated the Hamiltonian and diffeomorphism constraints with a scalar lapse N and a shift vector v^a. The expression for the Hamiltonian is only valid for single loops γ;

additional terms are present (see Ref. [22] for details) when one considers its action on multiloops.

The area derivative [23] is the natural differential operator that arises in loop space when one considers two loops to be topologically "close" if they differ by an infinitesimal element of area. The definition of the area derivative is $\Psi[\gamma\cdot\delta\gamma]=(1+\sigma^{ab}\Delta_{ab})\Psi[\gamma]$, where $\sigma^{ab}=\oint ds\,\delta\gamma^{[a}(s)\delta\dot{\gamma}^{b]}(s)$ is the element of area associated with the infinitesimal loop $\delta\gamma$ ($\gamma\cdot\delta\gamma$ stands for the composition of loops γ and $\delta\gamma$). We note that the introduction of this derivative in no way needs the introduction of a metric or other structure that can conflict with diffeomorphism invariance [22].

To calculate the action of the constraints on $\Psi[\gamma]$ we need the formulas for the area derivative of an X. Notice that the X in the invariant in question are always contracted with cyclic objects; therefore we only need their cyclic part, which we denote with a subscript c. One obtains

$$\Delta_{ab}(s)X_c^{a_1x_1\cdots a_nx_n}(\gamma)=\{X^{a_1x_1\cdots a_{n-1}x_{n-1}}(\gamma_s^s)2\delta^{a_n}_{[a}\partial_{b]}\delta(x_n-x)$$
$$-X^{a_1x_1\cdots a_{n-2}x_{n-2}}(\gamma_s^s)2\delta_a^{[a_{n-1}}\delta_b^{a_n]}\delta(x_{n-1}-x)\delta(x_n-x)\}_c\,, \qquad (7)$$

where $\gamma_s^s=\gamma_s^1\gamma_0^s$ and $x=x(s)$ (see, for instance, [24]). Note that this expression for the area derivative consists of two terms, one containing an X of one order less than the one of which the derivative is being taken, and one of two orders less. In what follows one also needs to take into account the following differential identities satisfied by the X's:

$$\partial_{a_i}(x_i)X_c^{a_1x_1\cdots a_ix_i\cdots a_nx_n}=\delta(x_{i-1}-x_i)X_c^{a_1x_1\cdots \not{a}_i\not{x}_i\cdots a_nx_n}-\delta(x_{i+1}-x_i)X_c^{a_1x_1\cdots \not{a}_i\not{x}_i\cdots a_nx_n}\,, \qquad (8)$$

where $x_{n+1}=x_1$ and $x_0=x_n$. The notation $\partial_{a_i}(x_i)$ means a partial derivative evaluated at x_i and our "generalized Einstein convention" does not apply to the x_i's inside the δ functions.

We will now indicate how one can explicitly show that the proposed wave function satisfies the diffeomorphism constraint. This is of interest in itself from the point of view of knot theory since it constitutes the first explicit proof that the expression is actually a knot invariant. (For reasons of space we cannot show the full calculation, which will appear in [20], but it can be completed straightforwardly with steps similar to the ones shown.)

Let us evaluate the action of the diffeomorphism constraint on ρ_1 for a single loop γ:

$$\hat{C}(v)\rho_1(\gamma)=\int ds\,v^d(p)\dot{\gamma}^e(s)h_{ax\,by\,cz}[X^{ax\,by}(\gamma)2\delta^c_{[d}\partial_{e]}\delta(p-z)-X^{ax}(\gamma)2\delta^b_{[d}\delta^c_{e]}\delta(p-y)\delta(p-z)]$$
$$=\int ds\,v^d(p)\dot{\gamma}^e(s)[2\partial_{[d}h_{ep]ax\,by}X^{ax\,by}(\gamma)-2h_{ax\,[dp\,ep]}X^{ax}(\gamma)]\,, \qquad (9)$$

where $p\equiv\gamma(s)$ and we have integrated by parts. Expanding the derivative on the $h_{ep\,ax\,by}$ and integrating by parts we get several terms. Some of them cancel using the differential identities for the X's and others combine with terms coming from the action of the constraint on ρ_2 to give a vanishing result. The application of the constraint to ρ_2 goes along the same lines as the calculation exhibited.

We now sketch the calculation for the Hamiltonian constraint. The Hamiltonian constraint is only nonvanishing when applied to intersecting loops. Here we compute the action of the Hamiltonian constraint evaluated for a single loop $\gamma=\gamma_1\gamma_2\gamma_3$ which intersects itself only at the point $x_{\rm int}$ where the γ_i's are combined. We assume the γ_i's not to have kinks, but γ is allowed to have kinks at the intersection. Such a loop with a triple self-intersection constitutes the generic case since there can be no more than three linearly independent tangent vectors at a point. Again we cannot give the complete calculation for reasons of space. There are three types of terms that arise, which are of order three, four, and five (here order is the number of tangent vectors). The terms of order three stem from the application of the constraint to ρ_1 and cancel among themselves. The terms of order four come from both ρ_1 and ρ_2 and cancel when combined. We now show explicitly the cancellation of the terms of order five. Evaluating the Hamiltonian constraint for the loop γ one obtains up to a factor Z depending on the regulator f_ϵ:

$$H(N)\Psi[\gamma_1\gamma_2\gamma_3]=ZN(x_{\rm int})\{\dot{\gamma}_3^a\dot{\gamma}_1^b\Delta_{ab}(1)\Psi[\gamma_3^{-1}\gamma_2^{-1}\gamma_1]+\text{cyc.}\}\,, \qquad (10)$$

where $\dot{\gamma}_i^a$ is the tangent at the intersection which is at the end of the path from $\gamma_i(0)$ to $\gamma_i(1)$, the area derivative acts at the end of the loop argument, and cyc. means cyclic permutations of the γ_i's. The fifth-order terms, which all arise from ρ_2, are

$$\{H(N)\Psi[\gamma]\}_{\text{5th order}}=ZN(x_{\rm int})\dot{\gamma}_3^a\dot{\gamma}_1^b2\delta^{a_4}_{[a}\partial_{b]}\delta(x_{\rm int}-x_4)X^{a_1x_1a_2x_2a_3x_3}(\gamma_3^{-1}\gamma_2^{-1}\gamma_1)g_{a_1x_1a_3x_3}g_{a_2x_2a_4x_4}+\text{cyc.}$$
$$=-2ZN(x_{\rm int})\dot{\gamma}_1^a\dot{\gamma}_2^b\dot{\gamma}_3^c\epsilon_{abc}g_{a_1x_1a_2x_2}[X^{a_1x_1}(\gamma_3)X^{a_2x_2}(\gamma_2)-X^{a_1x_1}(\gamma_3)X^{a_2x_2}(\gamma_1)]+\text{cyc.}\,, \qquad (11)$$

where we have integrated by parts to obtain the last equality. We notice that when all the cyclic terms are added the result is zero. (Actually the zero result is obtained in the limit where the regulator is removed. The factor Z is divergent in this limit and therefore one could consider a "renormalized" Hamiltonian constraint multiplying it by ϵ. This kind of situation appears for all other known solutions [2-5,21].)

The expression for the square root of the determinant of the three-metric in loop space can be computed following the same procedure outlined in Ref. [21] to calculate the action of the Hamiltonian constraint. The final result is

$$(\det q)^{1/2}(x)\Psi[\gamma_1\gamma_2\gamma_3] = -\tfrac{3}{2}Z\delta^3(x-x_{\rm int})\dot\gamma_1^a\dot\gamma_2^b\dot\gamma_3^c\epsilon_{abc}(\Psi[\gamma_1^{-1}\gamma_2\gamma_3]+\text{cyc.})\,, \tag{12}$$

and for the wave function $\Psi[\gamma]$ under consideration this expression is nonvanishing. This concludes the proof that $\Psi[\gamma]$ has properties as claimed.

Since the solution has a self-intersection at an isolated point, and it is at this point where the three-metric is nondegenerate, one could consider constructing a "weave" of loops with triple intersections at several points on the manifold. This could be a naive candidate for a "semiclassical" quantum state, which would have a three-metric nondegenerate in a set of discrete points and the net effect would be a nondegenerate metric averaged over large scales. The use of such wave functions has already been considered, in the context of a self-consistent perturbative scheme for the loop space representation by Ashtekar, Rovelli, and Smolin [25].

The generation of higher-order knot invariants and details of how these expressions are consistently framed will be discussed in a forthcoming publication [20].

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