# Effect of a Nonuniform Magnetic Field on a Two-Dimensional Electron Gas in the Ballistic Regime

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(Received 27 August 1991)

The single-particle electronic structure of a two-dimensional electron gas in a nonuniform magnetic field $B$ consists of states that propagate perpendicularly to the field gradient $\nabla B$ and have a remarkable time-reversal asymmetry. In one of the allowed directions the propagation has free-electron character and is confined to a narrow one-dimensional channel localized about the region where $|B|$ is minimum. In the opposite direction, the Landau states propagate throughout the rest of the sample with a velocity proportional to $|\nabla B|^{1/2}$.

PACS numbers: 72.15.Gd, 72.20.My, 73.20.Dx

The electrical behavior of a two-dimensional electron gas (2DEG) subjected to a uniform magnetic field has been studied recently in much detail [1,2]. In the ballistic regime, the system is characterized by stationary Landau states, which do not propagate and yield no contribution to its conductance. The charge flow observed in finite samples was shown to be due to edge states localized at the boundary of the 2DEG [3]. This Letter discusses the situation in which the magnetic field in the interior of the sample is not uniform. In this case the Landau states are no longer stationary but propagate perpendicularly to the field gradient and exhibit a remarkable time-reversal asymmetry. In one of the allowed directions the propagation has free-electron character, and it is confined to a narrow one-dimensional region localized near the line where the magnitude of the magnetic field is minimum. In the opposite direction the Landau states propagate throughout the rest of the sample with a velocity which depends on the field gradient.

The asymmetric propagation of electrons in a nonuniform magnetic field can be understood by the following classical argument illustrated in Fig. 1. Consider a 2DEG constrained by rigid walls to $-L/2 \le y \le L/2$, but infinite along the $x$ axis, and subjected to a magnetic field with a component perpendicular to the $x$-$y$ plane, given by

$$B(y) = B_1 y \,, \tag{1}$$

such that it changes sign at the line $y=0$ [4]. The trajectory of a classical electron moving in the plane has an instantaneous radius of curvature given by $r = cp/eB$, where $p$ is the momentum of the electron, and $B$ the local magnitude of the magnetic field [5]. This means that the part of the trajectory that scans larger (smaller) values of the magnetic field will have a smaller (larger) radius, giving rise to an open orbit that drifts perpendicularly to the field gradient, as illustrated by the drift ($d$) trajectories in Fig. 1. For the case of a sufficiently small field gradient, $\nabla B = B_1 \hat{\mathbf{y}}$, a straightforward classical calculation shows that the magnitude of the drift velocity is given by [5]

$$v = er^2 B_1/2mc \,. \tag{2}$$

Near the boundaries of the sample, collisions against the rigid walls force the electrons into skipping orbits as illustrated by the edge ($e$) trajectories in Fig. 1, which propagate in the same direction as the $d$ trajectories. On the other hand, in the region where $B \simeq 0$ the electrons scan magnetic fields with alternating positive and negative signs, giving rise to the $s$ trajectory, that propagates in a direction opposite to that of the $e$ and $d$ trajectories.

The confinement of the electron trajectories in a magnetic field is actually a purely quantum-mechanical effect, for in a classical description the radius of the electron orbits assumes a continuous sequence of values and can be arbitrarily large. For an electron in a uniform magnetic field $B = B_0$, the quantum-mechanical treatment is well known [6]. The solutions of the Schrödinger equation are Landau states, labeled by one of the components of the momentum (say $p_x$), and by a magnetic quantum number $n$. The corresponding wave functions have a spatial extension perpendicular to $\hat{\mathbf{x}}$ given by

$$\Delta y_n = [(n+\tfrac{1}{2})\hbar c/eB_0]^{1/2} \,, \tag{3}$$

and for a given $n$, they have all the same shape independ-

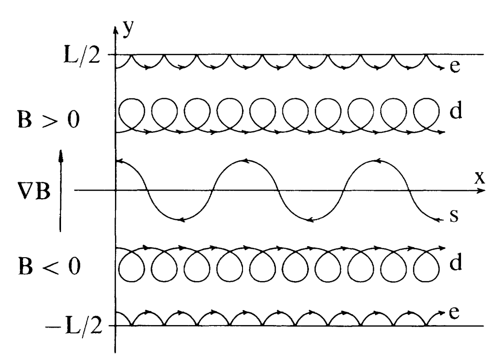


FIG. 1. Classical picture of the electron trajectories in a two-dimensional system constrained by rigid walls at $y = \pm L/2$ but infinite along the $x$ axis. The magnetic field $B = B_1 y$ is normal to the plane with a field gradient $\nabla B = B_1 \hat{\mathbf{y}}$, as shown in the picture. Trajectories $d$ and $e$ drift in the direction $+\hat{\mathbf{x}}$ while trajectory $s$, confined about the line where $B \simeq 0$, propagates in the direction $-\hat{\mathbf{x}}$.

ent of the momentum $p_x$, but are displaced with respect to one another in such a way that their centers are given by $y_0 = cp_x/eB_0$. This condition ensures that their velocity is $v_x = (1/m)(p_x - eB_0 y_0/c) = 0$, i.e., that the Landau states do not propagate. The momentum $p_x$ keeps track of the orthogonality of the Landau states, but it has no effect on their energy, which is given by

$$E_n = (n + \tfrac{1}{2}) e\hbar B_0/mc\,. \tag{4}$$

Equation (4) means that a two-dimensional electron gas in a uniform magnetic field is highly degenerate. In a finite geometry the Landau states with sufficiently large $p_x$ to be close to the boundary of the sample also have a larger energy, giving rise to a finite dispersion $E_n(p_x)$ and thus a velocity $v_x = \partial E/\partial p_x \neq 0$. These edge states can therefore contribute to charge flow [3].

The magnetic field enters in the Schrödinger equation as an additional momentum proportional to the vector potential. For the system of Fig. 1 the vector potential can be conveniently written as $\mathbf{A} = -\tfrac{1}{2} B_1 y^2 \hat{\mathbf{x}}$, from which the magnetic field (1) can be deduced using $B_z = -\partial A_x/\partial y$. The Hamiltonian of the system is given by

$$H = \frac{1}{2m}\left(p_x - \frac{eB_1}{2c} y^2\right)^2 + \frac{p_y^2}{2m}\,, \tag{5}$$

where $p_x, p_y$ are the components of the momentum. In the present work, the energy related to the spin degrees of freedom will be neglected, and that associated with the motion perpendicular to the $x$-$y$ plane will be assumed to be a constant [4]. Electron-electron interactions and impurity scattering will not be considered, but boundary scattering will be explicitly included in the solution of the Schrödinger equation through the boundary conditions for the wave function. The electric charge of the electron is given by $-e$. Since $[p_x, H] = 0$ we can write the wave function in the form $\psi = \chi(k_x, y)\exp(ik_x x)$, where $k_x = p_x/\hbar$, and $\chi(k_x, y)$ is a solution of the equation

$$\chi'' + (2m/\hbar^2)[E - V(k_x, y)]\chi = 0\,. \tag{6}$$

The effective potential

$$V(k_x, y) = \frac{1}{2m}\left(\hbar k_x - \frac{eB_1}{2c} y^2\right)^2, \tag{7}$$

incorporating the effect of the magnetic field, is illustrated in Fig. 2. Note the asymmetric dependence of $V(k_x, y)$ on $k_x$. The effective potential consists of a double well for $k_x > 0$, and of a single well shifted upwards in energy by $(\hbar k_x)^2/2m$ for $k_x < 0$. It is a symmetric function of $y$, with the single well centered at $y = 0$, the minima of the double wells located at

$$y_1 = \pm (2c\hbar k_x/eB_1)^{1/2} \quad (k_x > 0)\,, \tag{8}$$

and the value of the potential at the minima given by $V(k_x, y_1) = 0$.

In the present work, Eq. (6) was solved by expanding the wave function $\chi(k_x, y)$ in terms of oscillator functions (Hermite polynomials with a Gaussian factor), evaluating numerically the corresponding matrix elements of $V(k_x, y)$, and diagonalizing the resulting secular problem, for each $k_x$. In the numerical application the physical parameters have been chosen to correspond to the 2DEG in GaAs-AlGaAs heterostructures. The electron mass was taken to be $m = 0.067 m_e$, and the electron density per unit area was assumed to be $\rho_s = 4\times10^{-5}$ Å$^{-2}$. The Fermi vector is given by $k_F = (4\pi\rho_s/g_s)^{1/2} = 1.58\times10^{-2}$ Å$^{-1}$ (here $g_s = 2$ is the spin degeneracy), and the Fermi energy is $E_F = (\hbar k_F)^2/2m = 14.3$ meV. The configuration of Fig. 1 was chosen to keep the numerical effort to a minimum. The width of the sample was chosen to be $L = 2\times10^4$ Å, and for the magnetic-field gradient, a value $B_1 = 1$ G/Å was taken.

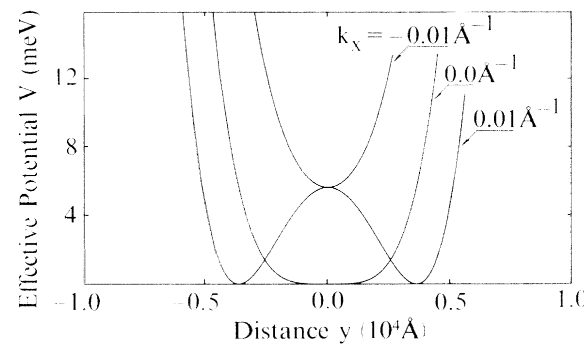


FIG. 2. Effective potential $V(k_x, y)$, characterizing the magnetic problem for $B = B_1 y$, with $B_1 = 1$ G/Å. It consists of a double well for $k_x > 0$, and of a single well shifted upwards in energy by $(\hbar k_x)^2/2m$ for $k_x < 0$.

The energy bands $E_n(k_x)$ shown in Fig. 3(a) exhibit a pronounced asymmetry as a function of $k_x$ that reflects the corresponding asymmetry of $V(k_x, y)$. For $k_x < 0$ the effective potential confines the electrons in the region where the magnetic field is $B \simeq 0$ (see Fig. 2). Accordingly, the energy bands characterizing the motion along $-\hat{\mathbf{x}}$ are free-electron-like. The band index $n$ labels the successive excited state of the magnetic problem (6). Note that for the excited states, the minima of the bands do not occur at $k_x = 0$, but at small positive values of $k_x$, such that for some states the momentum and the velocity have opposite signs. The reason for this effect is that for small positive values of $k_x$ the effective potential becomes wider, lowering the energies of the excited states, which become broader with increasing $n$.

For $k_x > 0$ the double well of the effective potential $V(k_x, y)$ splits the electron wave function [which is symmetrical (antisymmetrical) about $y = 0$, for $n$ = even (odd)] into two Landau states located at positions given by (8). When the wave vector $k_x$ becomes sufficiently large and the Landau states separate completely, symmetrical and antisymmetrical solutions have the same energy, and the energy bands become degenerate in pairs.

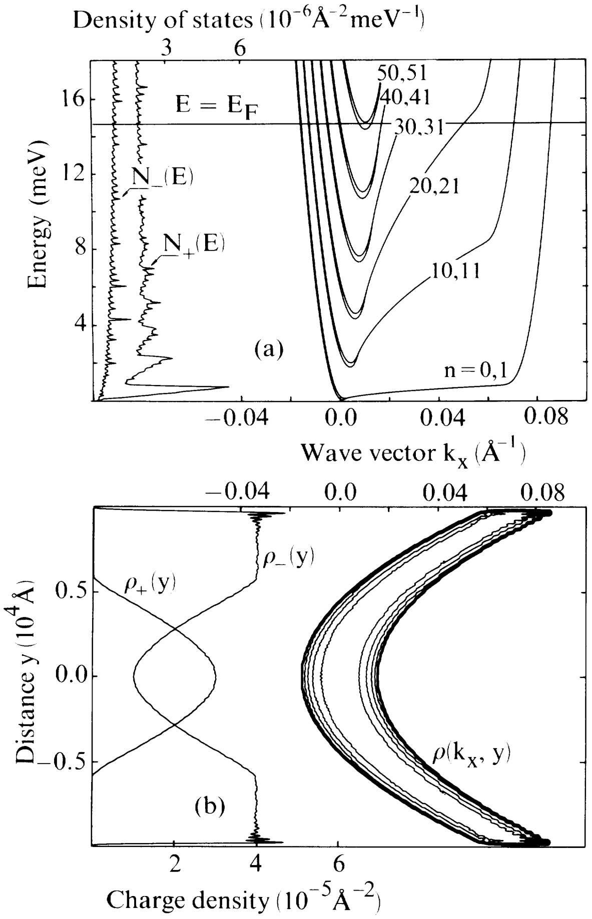


FIG. 3. (a) Energy bands $E_n(k_x)$ for some values of $n$, and densities of states $N_+(E)$ and $N_-(E)$ for electrons with $v_x>0$ and $v_x<0$, respectively. (b) Charge densities $\rho_+(y)$ and $\rho_-(y)$ for electrons with $v_x>0$ and $v_x<0$, respectively. Also shown is the spectral charge density $\rho(k_x,y)$. The contour lines represent densities given by $6l\times10^{-4}\,e\,\text{Å}^{-1}$, for $l=1,\ldots,10$.

Note that in the region $v_x>0$, the energy bands are flatter than for $v_x<0$, because of the energy lowering in the magnetic field. The energy bands are well represented by the expression

$$E_n(k_x)\simeq(n+\tfrac{1}{2})\frac{\hbar}{m}\left(\frac{eB_1\hbar k_x}{2c}\right)^{1/2}\quad(v_x>0)\,,\tag{9}$$

which follows from (4) using $B_0\simeq B_1y_1$, with $y_1$ given by (8). The dispersion in the energy bands arises because the Landau states with increasing momentum are pushed into regions of larger magnetic field [cf. Eq. (8)], and larger energy. From (9) the velocity of the Landau states is

$$v_x=\hbar^{-1}\frac{\partial E_n}{\partial k_x}\simeq(n+\tfrac{1}{2})\frac{\hbar}{m|y_1|}\,,\tag{10}$$

which agrees with the classical value (2), if one uses $r^2=2(\Delta y_n)^2$, where $\Delta y_n$ is given by Eq. (3) with $B_0=|y_1|B_1$.

The finite drift velocity in the interior of a 2DEG subjected to a magnetic field is due to the transverse energy dependence of the Landau states, which can also be achieved by electrostatic means. For instance, the parabolic confinement of a 2DEG in a narrow channel leads, in a uniform magnetic field, to a finite mobility of the Landau states along the channel with an increased effective mass [7]. Another situation where a transverse energy variation of the Landau states takes place was reported by Weiss *et al.*, who studied the combined effect of a weak periodic potential and a uniform magnetic field on a 2DEG [8]. The observed dependence of the current flow parallel to the equipotentials on the strength of the magnetic field was explained by various authors in terms of electrons drifting in the interior of the sample [9–12]. However, as long as the electrostatic energy variation perpendicular to the electron drift is symmetric, the behavior of the system under a time-reversal operation is also symmetric. Recently, Kouwenhoven *et al.* performed an experiment where the magnetic field does actually break the time-reversal symmetry [13]. They studied the effect of a uniform magnetic field on a 2DEG in a narrow channel subjected to a potential that was periodic along the channel, but with an asymmetric lateral confinement. The resulting electronic structure exhibits skew minibands [2], that resemble those of Fig. 3(a).

When the momentum becomes sufficiently large and positive that $y_1\simeq L$ [cf. Eq. (8)], the steepness of the energy bands increases abruptly, becoming comparable to that of the free-electron ones. This part of the spectrum describes the edge states, well known in the context of the quantum Hall effect [3]. In addition to the opposite signs of their velocity, there are two essential differences between the edge states and the free-electron states that occur in the interior of the sample. First, the inner states involve only magnetic scattering. Second, they lie lowest in energy and are always occupied, while the edge states can be empty if the magnetic field becomes sufficiently large that only the band $n=0$ is below the Fermi level.

The partial densities of states $N_+(E)$ and $N_-(E)$ for electrons with $v_x>0$ and $v_x<0$, respectively, are also shown in Fig. 3(a). In the region where (10) is applicable, the contribution of the individual bands to the density of states is proportional to the energy $E$, as is apparent in the sawtooth form of $N_+(E)$. On the other hand, $N_-(E)$ is clearly free-electron-like. The total density of states $N(E)=N_+(E)+N_-(E)$ oscillates about the average value given by the two-dimensional free-electron value $g_sm/2\pi\hbar^2=2.8\times10^{-6}$ Å$^{-2}$meV$^{-1}$. Figure 3(b) shows that the spectral charge density, given by $\rho(k_x,y)=\sum_n|\chi_n(k_x,y)|^2$ (here the sum includes all the energies below the Fermi), is roughly constant over the variable range where

$$V(k_x,y)\le E_F\,.\tag{11}$$

Also shown in Fig. 3(b) are the charge densities $\rho_+(y)$

and $\rho_-(y)$ for electrons with $v_x > 0$ and $v_x < 0$, respectively. Note that $\rho_+$ remains confined within a one-dimensional region of width $w \simeq 10^4$ Å, forming a channel in which charge flow in the direction $-\hat{\mathbf{x}}$ takes place. It follows from (7) and (11) that this width depends on the magnetic field according to $w \simeq B_1^{-1/2}$. The charge flow in the opposite direction takes place throughout the rest of the sample, including the edge states. Of course, the net current in equilibrium is $j_n = -e\int v_n(k_x)dk_x = 0$, for each band. It is also interesting to point out that the total electron density $\rho(y) = \rho_+(y) + \rho_-(y)$ is uniform across the sample, i.e., no charging effects take place.

According to the theory put forward by Landauer [14] and Büttiker [3], the conductance $G$ is proportional to the number of propagating modes at the Fermi energy, which for our system is equal to the number of bands with energies below $E_F$, given by $n_{\max} = (w/\pi)k_F$. In the absence of the magnetic field or for small field values, $w = L$, i.e., the conductance is proportional to the width of the sample, as expected. With increasing magnetic field $w < L$, and the conductance decreases, remaining proportional to $w \simeq B_1^{-1/2}$ (see above). Another important consequence of the theory of Landauer and Büttiker is that the conductance is the same in the positive and negative $x$ directions, because each energy band cuts the Fermi energy in two points, one with $v_x > 0$ and another with $v_x < 0$. It may be interesting to test this statement experimentally, since states moving in the positive and negative directions have very different velocities and in a nonideal 2DEG will involve different scattering processes.

The main result of this work is that in the presence of a nonuniform magnetic-field transport properties become one dimensional. Charge flow takes place only in the direction perpendicular to the field gradient with a conductance proportional to $|\nabla B|^{1/2}$. These results are not restricted to the geometry of Fig. 1, which was chosen to simplify the presentation. The present study can be trivially extended to the case where $B(y) = B_0 + B_1 y$, representing a general nonuniform magnetic field. The effective potential will become an asymmetric function of $y$, and all degeneracies in the energy bands will be removed. The minimum of the energy bands will occur now for a value of the momentum $k_x = k_{x0} < 0$, even for the magnetic ground state, $n = 0$. However, Eqs. (9) and (10) will remain valid with $k_x$ referred to $k_{x0}$, and propagation in the $+\hat{\mathbf{x}}$ direction will be confined about the line where $B(y)$ is minimum. Experimentally, such a field could be generated by covering part of the sample (maybe on both sides) with a superconducting film before submerging it in a homogeneous field. Since the superconductor shields the magnetic field there will be a large field gradient at the boundary between the covered and uncovered parts.

The author gratefully acknowledges stimulating discussions with J. R. Manson and D. L. Mills.

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