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Parity Conservation in Chern-Simons Theories and the Anyon Interpretation

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It is shown that by including both the spin up and spin down components of a spin one-half field, a parity conserving Chern-Simons theory can be constructed. For a single fermion field the model contains (aside from the mass and a coupling constant) a parameter ϵ which can have the values ± 1. It is shown that only the positive value allows an interpretation in terms of anyons. This result thus serves to sharpen the frequently overlooked distinction between the Chern-Simons interaction and the anyon. While the former can be introduced into any $(2+1)$-dimensional field theory, the latter is seen to be a special limiting case obtained only when relativistic and spin considerations can be ignored.

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The introduction [1] of the Chern-Simons term $\epsilon_{\mu\nu\alpha}A^\mu\partial^\alpha A^\nu$ into the QED Lagrangian in $2+1$ space was remarkable in that it demonstrated the possibility of having a finite photon mass at the free-field level in a gauge invariant theory. It was subsequently shown [2] that rather interesting results can be obtained even when the Chern-Simons term is taken to be the entire gauge field Lagrangian. In particular such a system lacks an elementary photon so that the sole effect of the gauge field is to mediate long-range interactions between charged particles. It is also fairly straightforward to show [3] that in the Galilean (or "nonrelativistic") limit the two-particle sector of this model is equivalent to the Aharonov-Bohm effect [4].

Since high-T_c superconductivity is widely thought to be a planar phenomenon, this photonless gauge theory has been the object of considerable application in that context. There are, however, two commonly accepted views to be found in such work which require careful scrutiny. These are as follows: (1) "The presence of a Chern-Simons term implies that parity P and time reversal T are violated, of course" [5]. One also encounters statements to the effect that [6] "in T-invariant systems, the only possible statistics are those corresponding to bosons and fermions." (2) Equivalence is assumed between Chern-Simons and fractional statistics theories (i.e., the "anyon" [7]). In fact the first point (which is trivially true in the simplest Chern-Simons theory) has motivated a number [8–10] of experiments designed specifically to detect such P and T violating effects. The results obtained to date are, however, inconclusive on this point.

Before proceeding it is important to sharpen the issues involved in point (1) as detailed above. Applications to date of fractional statistics to condensed matter systems implicitly assume the simplest possible formulation of this theory. That model could indeed be contradicted by experiments such as those described in Refs. [8–10]. On the other hand, null results for such tests would not exclude parity conserving theories of the type constructed in this paper. It will also be shown that these latter models are sometimes compatible with fractional statistics and sometimes not.

It must also be noted that the possibility of restoring P and T invariance to Chern-Simons theories has been mentioned in the literature [1,11]. In fact there is general recognition of the fact that the formal device of parity doubling allows the restoration of parity invariance to any given theory. This paper enlarges this issue somewhat by carrying out the explicit construction of the most general Chern-Simons parity invariant theory. This provides the specific framework necessary to discuss the crucial issue concerning the equivalence or nonequivalence of Chern-Simons and fractional statistics theories.

To carry out the construction one notes that for a set of gauge fields ϕ_i^μ $(i=1,2,\ldots,N)$ the most general Abelian Chern-Simons Lagrangian can be taken as

$$\mathcal{L}_{0g}=\tfrac{1}{2}\,\phi^\mu\epsilon_{\mu\nu\alpha}A\partial^\alpha\phi^\nu\,,$$

where A is a real symmetric matrix. An orthogonal transformation can be used to diagonalize A while a subsequent rescaling of the fields yields

$$\mathcal{L}_{0g}=\tfrac{1}{2}\sum_i s_i\phi_i^\mu\epsilon_{\mu\nu\alpha}\partial^\alpha\phi_i^\nu\,,$$

where $s_i = \pm 1$. The requirement that $\mathcal{L}_{0g}$ be parity invariant implies that N must be even and that the degeneracies of $s_i = +1$ and $s_i = -1$ are equal. Thus one writes

$$\mathcal{L}_{0g} = \frac{1}{2} \sum_{i=1}^{n} [\phi^{\mu}_{i+} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{i+} - \phi^{\mu}_{i-} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{i-}] , \qquad (1)$$

where the parity transformation of the fields is expressed by

$$P\phi^{\mu}_{i\pm}(x,t)P^{-1} = \phi^{\mu}_{i\mp}(\tilde{x},t) \times \begin{cases} +1, & \mu = 0,1, \\ -1, & \mu = 2, \end{cases}$$

and $\tilde{x}_i = (x_1, -x_2)$.

For a spin one-half (two component) field of mass M and spin component $s/2$ the free Lagrangian can be written as

$$\mathcal{L}_{0f} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - M\bar{\psi}\psi , \qquad (2)$$

where the Dirac matrices can be taken to have the Pauli form [12]

$$\beta = \sigma_3, \quad \beta\gamma^i = (\sigma_1, s\sigma_2) . \qquad (3)$$

A theory in which both spin up ($s = +1$) and spin down ($s = -1$) particles are accommodated is then trivially realized by replacing s in (3) by the third Pauli matrix ρ_3 (where $[\rho_i, \sigma_j] = 0$) and a corresponding doubling of the number of components of ψ. The parity operation for the ψ field is then seen to be

$$P\psi(x,t)P^{-1} = \rho_1 \psi(\tilde{x},t) ,$$

which is clearly an invariance of the Lagrangian (2).

The most general gauge coupling between the set ϕ^{μ} and the spinor field ψ is

$$\mathcal{L}_{\text{int}} = \sum_{i=1}^{n} [\bar{\psi}\gamma_{\mu} P_{+} \psi (e_i \phi^{\mu}_{i+} + g_i \phi^{\mu}_{i-}) + \bar{\psi}\gamma_{\mu} P_{-} \psi (e_i \phi^{\mu}_{i-} + g_i \phi^{\mu}_{i+})] , \qquad (4)$$

where the projectors $P_{\pm}$ onto the spin up and spin down fields are

$$P_{\pm} = \frac{1}{2}(1 \pm \rho_3) .$$

At this point it is straightforward to establish that since only a single linear combination of the ϕ^{μ}_{i+} and of the ϕ^{μ}_{i-} appears in (4), a redefinition of the fields allows one to set $n = 1$ (the remaining gauge fields $\phi^{\mu}_{i\pm}$ in fact reduce to zero for $i > 1$). Thus upon combining (1), (2), and (4) one obtains as the total Lagrangian

$$\mathcal{L} = \frac{1}{2}(\phi^{\mu}_{+} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{+} - \phi^{\mu}_{-} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{-}) + i\bar{\psi}\gamma_{\mu}[P_{+}(\partial^{\mu} - ie\phi^{\mu}_{+} - ig\phi^{\mu}_{-}) + P_{-}(\partial^{\mu} - ie\phi^{\mu}_{-} - ig\phi^{\mu}_{+})]\psi - M\bar{\psi}\psi \qquad (5)$$

of the general parity conserving theory.

While not immediately apparent, it turns out that the Lagrangian (5) can be reduced from a theory containing two coupling constants e and g to one involving only a single arbitrary constant λ and a parameter ϵ which can take the values ± 1. To demonstrate this one makes the replacements

$$\phi^{\mu}_{\pm} \rightarrow \kappa |e^2 - g^2|^{-1/2} (e\phi^{\mu}_{\pm} - g\phi^{\mu}_{\mp}) ,$$

where $\kappa^2 = 1$, to bring (5) to the form

$$\mathcal{L} = \frac{1}{2}(\phi^{\mu}_{+} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{+} - \phi^{\mu}_{-} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{-}) \frac{e^2 - g^2}{|e^2 - g^2|}$$

$$+ i\bar{\psi}\gamma_{\mu}\left[P_{+}\left(\partial^{\mu} - i\kappa \frac{e^2 - g^2}{|e^2 - g^2|^{1/2}} \phi^{\mu}_{+}\right) + P_{-}\left(\partial^{\mu} - i\kappa \frac{e^2 - g^2}{|e^2 - g^2|^{1/2}} \phi^{\mu}_{-}\right)\right]\psi - M\bar{\psi}\psi . \qquad (6)$$

A minor relabeling of fields and coupling constants now establishes the equivalence of (6) to the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\phi^{\mu}_{+} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{+} - \phi^{\mu}_{-} \epsilon_{\mu\nu\alpha} \partial^{\alpha} \phi^{\nu}_{-}) + i\bar{\psi}\gamma_{\mu}[P_{+}(\partial^{\mu} - i\lambda\phi^{\mu}_{\epsilon}) + P_{-}(\partial^{\mu} - i\lambda\phi^{\mu}_{-\epsilon})]\psi - M\bar{\psi}\psi , \qquad (7)$$

with λ a coupling constant and ϵ a parameter which is $+1$ (-1) for the situation in which the spin up field is coupled to the gauge field with positive (negative) Chern-Simons coefficient and the spin down field is coupled to the gauge field with negative (positive) Chern-Simons coefficient. Thus (7) is seen to be the most general gauge and parity invariant Chern-Simons theory for a mass-M fermion. Its construction furthermore establishes that a failure to detect experimentally the breakdown of parity invariance does not allow one to infer the absence of Chern-Simons terms in the system.

The equations of motion implied by (7) have the form

$$\gamma_{\mu}[(1/i)\partial^{\mu} - \lambda P_{+}\phi^{\mu}_{\epsilon} - \lambda P_{-}\phi^{\mu}_{-\epsilon}]\psi + M\psi = 0 , \qquad (8)$$

$$\epsilon_{\mu\nu\alpha}\partial^{\nu}\phi^{\alpha}_{\pm\epsilon} = \pm\epsilon\lambda\bar{\psi}\gamma^{\mu}P_{\pm}\psi . \qquad (9)$$

Using the results of Ref. [2] one infers from (9) that the fields $\phi^{\mu}_{\pm\epsilon}$ are expressible in the radiation gauge ($\nabla \cdot \phi_{\pm\epsilon} = 0$) as nonlocal functions of the current operators. Specifically

$$\phi^{i}_{\pm\epsilon}(x,t) = \mp\epsilon\lambda\epsilon_{ij}\nabla_j \int d^2x' \mathcal{D}(x - x') \times \psi^{\dagger}(x',t)P_{\pm}\psi(x',t) , \qquad (10)$$

$$\phi^{0}_{\pm\epsilon}(x,t) = \mp\epsilon\lambda \int d^2x' \bar{\psi}(x',t)\gamma P_{\pm}\psi(x',t) \times \nabla'\mathcal{D}(x - x') , \qquad (11)$$

where $\mathcal{D}(x)$ is defined by

$$-\nabla^2\mathcal{D}(x) = \delta(x) .$$

One of the most interesting aspects of the gauge theory of Ref. [2] was the appearance of a noncanonical contribution to the angular momentum which is proportional to

Q^2 where Q is the charge operator. In the model under consideration here the corresponding term is $\epsilon(\lambda^2/4\pi)\times(Q_+^2-Q_-^2)$, where

$$Q_\pm \equiv \int d^2x\psi^\dagger P_\pm\psi .$$

This immediately yields for the commutator of the angular momentum J with $\psi_\pm(x)$ the result

$$[J,\psi_\pm(x)] = -\left(r\times\frac{1}{i}\nabla \pm \frac{1}{2}\sigma_3\right)\psi_\pm(x) \mp \epsilon\frac{\lambda^2}{4\pi}\{Q_\pm,\psi_\pm(x)\}$$

and shows that anomalous spin effects can also occur in parity invariant theories.

In order to study the implications with regard to fractional statistics it is necessary to pass to the Galilean limit of the model. Since the Galilean version of the photonless gauge theory has been developed in some detail in a previous work [13], it is fairly straightforward to obtain the desired result. Noting that in that limit $\phi^\mu \to (\phi,\vec{\phi})$ and using the spin one-half results of Levy-Leblond [14] one finds

$$\mathcal{L} = -\frac{1}{2}\phi_+\nabla\times\phi_+ - \frac{1}{2}\phi_+\times\nabla\phi_+ - \frac{1}{2}\phi_+\times\frac{\partial}{\partial t}\phi_+ + \frac{1}{2}\phi_-\nabla\times\phi_- + \frac{1}{2}\phi_-\times\nabla\phi_- + \frac{1}{2}\phi_-\times\frac{\partial}{\partial t}\phi_-$$
$$+\psi^\dagger\left[\left(i\frac{\partial}{\partial t} - \lambda P_+\phi_\epsilon - \lambda P_-\phi_{-\epsilon}\right)\frac{1}{2}(1+\sigma_3) + M(1-\sigma_3) - \sigma_1(-i\nabla_1 - \lambda P_+\phi_\epsilon^1 - \lambda P_-\phi_{-\epsilon}^1)\right.$$
$$\left. -\rho_3\sigma_2(-i\nabla_2 - \lambda P_+\phi_\epsilon^2 - \lambda P_-\phi_{-\epsilon}^2)\right]\psi . \qquad (12)$$

The Hamiltonian which follows from (12) is found to have the surprisingly simple structure [15]

$$H = H_+ + H_- , \qquad (13)$$

$$H_\pm = \frac{1}{2M}\int d^2x[\psi_\pm^\dagger(\vec{p} - \lambda\vec{\phi}_{\pm\epsilon})^2\psi_\pm + \epsilon\lambda^2(\psi_\pm^\dagger\psi_\pm)^2] , \qquad (14)$$

where the $\vec{\phi}_{\pm\epsilon}$ are given by (10) and the operators $\psi_\pm$ are the components of ψ projected by $\frac{1}{2}(1+\sigma_3)P_\pm$, i.e.,

$$\psi_\pm \equiv \tfrac{1}{4}(1+\sigma_3)(1\pm\rho_3)\psi .$$

The latter can be seen to satisfy the equal-time commutation relations

$$\{\psi_\pm(x,t),\psi_\pm^\dagger(x',t)\} = \delta(x-x') ,$$
$$\{\psi_\pm(x,t),\psi_\mp^\dagger(x',t)\} = 0$$

and clearly comprise a parity doublet in that

$$P\psi_\pm(x,t)P^{-1} = \psi_\mp(\tilde{x},t) .$$

It is important to note that because of (13) and (14) the Hilbert space of states factors into a product of spin up particle states times a corresponding set of spin down particle states. Thus it is sufficient to consider the eigenvalue problem

$$H|N_\pm\rangle = E|N_\pm\rangle ,$$

where the state $|N_\pm\rangle$ is a state of N_+ spin up or N_- spin down particles. It is straightforward to show that the vacuum and single-particle states are trivial since the Hamiltonian acting on the state

$$|1_\pm\rangle = \int d^2x f_\pm(x)\psi_\pm^\dagger(x)|0\rangle$$

implies a free Schrödinger equation for $f_\pm(x)$. In the case of the two-body state, however, one obtains from the eigenvalue equation for

$$|2_\pm\rangle = \int d^2x_1 d^2x_2 f_\pm(x_1,x_2)\psi_\pm^\dagger(x_1)\psi_\pm^\dagger(x_2)|0\rangle ,$$

where

$$f_\pm(x_1,x_2) = -f_\pm(x_2,x_1) ,$$

that

$$Ef_\pm(x_1,x_2) = \frac{1}{2M}\{\Pi_\pm^{(1)2} + \Pi_\pm^{(2)2} + 2\epsilon\lambda^2\delta(x_1-x_2)\} \times f_\pm(x_1,x_2) ,$$

where

$$\Pi_\pm^{(1,2)} \equiv \frac{1}{i}\nabla^{(1,2)} \mp \epsilon\frac{\lambda^2}{4\pi}\tilde{\nabla}^{(1,2)}\ln|r_1-r_2|^2$$

and

$$(\tilde{\nabla})_i \equiv \epsilon_{ij}\nabla_j .$$

Upon separating the center-of-mass coordinate one derives total momentum eigenfunctions

$$f_\pm(x_1,x_2) = \exp\{(i/2)P\cdot(x_1+x_2)\}g_\pm(x_1-x_2) ,$$

where

$$k^2 g_\pm(x_1-x_2) \equiv \left(E - \frac{P^2}{4M}\right)Mg_\pm(x_1-x_2) = \left[-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{i}\frac{\partial}{\partial\phi} \pm \frac{\epsilon\lambda^2}{2\pi}\right) + \epsilon\frac{\lambda^2}{2\pi}\frac{1}{r}\delta(r)\right]g_\pm(r,\phi) ,$$

with r and ϕ denoting polar coordinates of $x_1 - x_2$. Upon taking $g_\pm(r,\phi) = e^{im\phi} g_\pm(r)$ there follows

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + k^2 - \frac{1}{r^2}(m+\alpha_\pm)^2 \mp \alpha_\pm \frac{1}{r}\delta(r)\right] g_\pm(r) = 0\,, \tag{15}$$

where $\alpha_\pm \equiv \pm \epsilon\lambda^2/2\pi$. This result is recognizable as the equation for the Aharonov-Bohm effect of spin one-half particles with spin projections $s = \pm 1$ and flux parameter $\alpha_\pm$. What is known about this system is that it has singular solutions when the coefficient of the delta function gives an attractive effect [12]. In the current context this means the case $\epsilon = -1$.

The implications of this result for fractional statistics and the anyon are now immediate. For $\epsilon = +1$ the effect of the delta function is repulsive and it is known from Ref. [12] that such a term implies the usual Aharonov-Bohm amplitude. This is precisely the familiar situation in which fractional statistics can be used to replace the Chern-Simons interaction and one thus concludes that an anyon interpretation is consistent. On the other hand, since the $\epsilon = -1$ case has singular solutions (for a specific value of m), one cannot in this case eliminate the Aharonov-Bohm interaction by a singular gauge transformation and a corresponding discontinuity condition on the wave function across a singularity line. Thus the anyon is not a consistent interpretation of the Chern-Simons theory in the case $\epsilon = -1$.

A striking illustration of the inadmissibility of the anyon interpretation when $\epsilon = -1$ is provided by a calculation of the second virial coefficient for a system of spin up or spin down particles. It is straightforward using techniques of Ref. [16] to obtain for the spin down case the virial coefficient $B_2(\alpha_-, T)$ at temperature T. In terms of the thermal wavelength $\lambda_T = (2\pi MkT)^{1/2}$,

$$B_2(\alpha_-,T) = \tfrac{1}{4}\lambda_T^2 \begin{cases} 1 - 2\beta^2, & N \text{ even}, \\ 1 - 2(\beta-1)^2, & N \text{ odd},\ N < 0, \\ 1 - 2(\beta+1)^2, & N \text{ odd},\ N > 0\,, \end{cases}$$

where $\alpha_- = N + \beta$ with N an integer and $0 \le \beta < 1$. The result is plotted in Fig. 1. It is sufficient to note that the discontinuities in B_2 and the nonperiodicity in α are both at variance with the anyon interpretation.

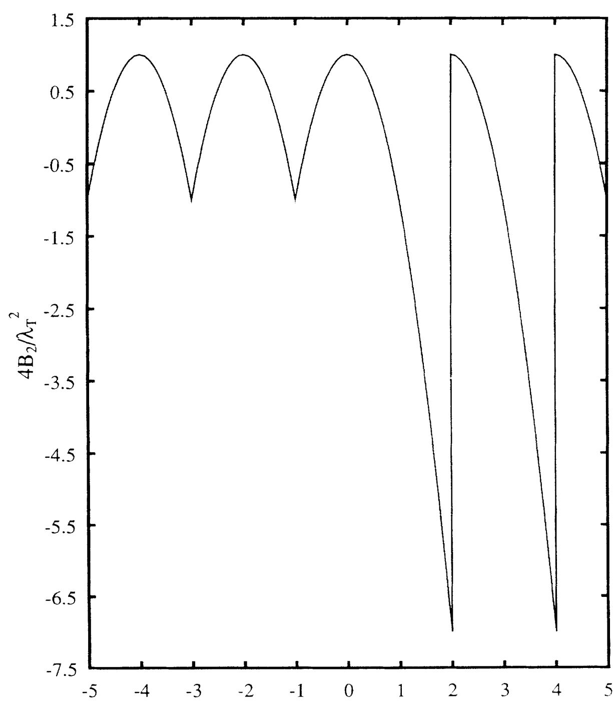

FIG. 1. The virial coefficient $B_2(\alpha_-, T)$ for the spin down case.

While this establishes that fractional statistics cannot generally be inferred in a Chern-Simons gauge theory, one might nonetheless ask whether fractional statistics could be considered "more fundamental" than Chern-Simons dynamics. Such a view is difficult to maintain, particularly when viewed in the light of what is known about the dynamics of spin. Since, for example, helicity is conserved for a Dirac particle moving in a time-independent magnetic field (a result which underlies $g-2$ experiments), the rejection of the Chern-Simons approach (which respects this helicity conservation) in favor of fractional statistics (which does not [17]) contradicts well established experimental results. Thus the use of fractional statistics as a physical postulate in applications other than the nonrelativistic treatment of spinless particles must certainly be considered suspect. Chern-Simons gauge theories, on the other hand, have no such limitations and clearly include fractional statistics as a special case in all instances where the latter can be successfully applied. If one subscribes to the philosophical principle that the assumptions introduced to explain a phenomenon should not be increased unnecessarily (Occam's razor), the anyon must seem a superfluous concept at best.

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