Universal Conductance Fluctuations in the Presence of Landau Quantization

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We generalize the analytic theory of universal conductance fluctuations to systems with Landau-level quantization. Results are valid to leading order in $1/g$ (g is conductance) but for arbitrary magnetic field. We show that the field only enters through the diffusion constant and cancels in the variance of g, which hence remains $\sim(e^2/h)^2$ over the entire range of magnetic fields. However, the correlation range B_c does vary with field in good agreement with experiments.

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Although the theory of mesoscopic fluctuation phenomena has been well developed in recent years, very little work has been done relating to two-dimensional conductors in high magnetic field, a system of great experimental interest. It has been observed that resistance fluctuations do occur in such devices as a function of magnetic field up to arbitrarily large fields [1,2]. At intermediate fields they are superimposed on the Shubnikov–de Haas oscillations (which indicate the presence of strong Landau quantization), and at higher fields (typically $B>4$ T) they occur in transition regions between plateaus in ρ_{xy} and zeros in ρ_{xx}. It has also been widely observed [1,2] that B_c, the typical field scale of the fluctuations (spacing of the features), increases with magnetic field. The transport fluctuations of two- and three-dimensional conductors in fields which are weak enough that the cyclotron radius r_c is larger than elastic mean free path l is well described by the conventional theory of universal conductance fluctuations (UCF) [3,4] based on perturbation theory in $(k_f l)^{-1}$. In this regime one finds that the variance of the conductance Var(g) for a phase-coherent conductor is $\sim(e^2/h)^2$ (independent of the degree of disorder) and the spacing of the features is independent of B. However, the regime of strong Landau-level (LL) quantization encountered in 2D conductors is not described by this theory, which assumes $r_c \gg l \rightarrow \omega_c\tau \ll 1$ (although some numerical results exist [5]). The experiments cited above violate this condition by as much as 2 orders of magnitude, and show significant differences from the standard behavior. It is thus an open question which if any aspects of the conventional theory survive strong LL quantization.

Here we present results from a perturbative analytic treatment of the conductance fluctuations of two-dimensional systems in high magnetic field which finds that as long as the motion at large distances is diffusive Var(g) remains independent of the degree of disorder and magnetic field. Our results are based on the self-consistent Born approximation (SCBA) introduced by Ando and Uemura [6], and shown by Carra, Chalker, and Benedict [7] to be the leading term in a systematic expansion in $1/N$ (where N indexes the LL at ε_f) when the disordered potential is short ranged. Generalizing the results of Carra, Chalker, and Benedict we find the SCBA to generate the leading term in an expansion in $1/\bar{g}$, where $\bar{g}\sim N$ for $r_c \ll l$ [6] and $\bar{g}\sim k_f l$ for $l \ll r_c$. Our results shed light on previous high-field calculations of the weak localization [7,8] and Coulomb [9] corrections to $\bar{g}$ in the unitary ensemble. In both cases the form of the quantum interference correction was found to be independent of the strength of the LL splitting but the origin of this field independence was unclear. Here we show that within the SCBA quite generally the field only enters through a B-dependent diffusion constant. A consequence is that quantities such as the conductance fluctuations and interaction correction which are independent of the diffusion constant are strictly independent of field (once time-reversal symmetry is broken). Because the correlation ranges E_c and B_c which determine the scale of the fluctuations do depend on the diffusion constant, they are found to vary with magnetic field. We evaluate the field dependence of B_c below and compare it to the experiments of Ref. [2], finding good agreement.

As in the standard theory [3,4], the leading perturbative contribution to Var(g) neglects localization effects; in addition, it is known that perturbation theory cannot describe the extended states in the center of the LL at infinite system size [10]. However, for mesoscopic systems (in which the localization length often may exceed the system dimensions) we expect our results to give a reasonable description of the behavior near the center of the LL of a mesoscopic two-dimensional electron gas. Moreover, our generalization of the perturbative approach should allow extension of the calculations of mesoscopic fluctuations of other physical quantities to the regime $\omega_c\tau \gg 1$.

We study a model of noninteracting electrons confined to two dimensions moving in a uniform perpendicular magnetic field under the influence of a weak short-range disordered potential $V(\mathbf{r})$. We take $V(\mathbf{r})$ to have zero mean value and white-noise statistics:

$$\langle V(\mathbf{r})V(\mathbf{r}')\rangle = c_i u^2 \delta(\mathbf{r}-\mathbf{r}'), \tag{1}$$

where c_i is the impurity density and u^2 is the scattering strength. In the SCBA for this model, the self-energy is also short ranged, $\Sigma(\mathbf{r},\mathbf{r}') = c_i u^2 \bar{G}(\mathbf{r},\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')$, where $\bar{G}$ is the full disorder-averaged one-particle Green function.

Hence $\bar{G}(\mathbf{r},\mathbf{r}')$ satisfies the self-consistent equation

$$[E-c_i u^2\bar{G}(\mathbf{r},\mathbf{r})-H_0]\bar{G}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')\,. \tag{2}$$

Since on average the system is translationally invariant, $c_i u^2\bar{G}(\mathbf{r},\mathbf{r},E)\equiv\Sigma(E)$ is just a complex function of energy. It immediately follows that

$$\bar{G}(\mathbf{r},\mathbf{r}',E)=\sum_{n=0}^{\infty}\frac{P_n(\mathbf{r},\mathbf{r}')}{E-E_n-\Sigma(E)}\,, \tag{3}$$

where $P_n(\mathbf{r},\mathbf{r}')$ is the projection operator onto the nth LL in real space and $E_n=(n+\frac{1}{2})\hbar\omega_c$. P_n is known analytically [7], but here we need only use the fact that the modulus of P_n decays on the scale of $r_c=\sqrt{n+1/2}\lambda$ [where the magnetic length $\lambda=(\hbar c/eB)^{1/2}$] and its phase $\phi(\mathbf{r},\mathbf{r}')=(e/\hbar c)\mathbf{A}(\mathbf{r})\cdot(\mathbf{r}-\mathbf{r}')$. Setting $\mathbf{r}=\mathbf{r}'$ in Eq. (3) yields a self-consistent equation for $\Sigma(E)$. In the high-field limit, in which the LL broadening $v\ll\hbar\omega_c$, the LL at the Fermi surface dominates in Eq. (3) and $\bar{G}(\mathbf{r},\mathbf{r}')\sim P_N(\mathbf{r},\mathbf{r}')$, showing $\bar{G}(\mathbf{r},\mathbf{r}')$ has a decay length r_c. Setting $\mathbf{r}=\mathbf{r}'$ in Eq. (3) allows one to obtain [6] $v=(c_i u^2/2\pi\lambda^2)^{1/2}=(\hbar^2\omega_c/\tau)^{1/2}$. In the limit $B\to 0$ the self-consistency condition on $\bar{G}$ may be neglected to leading order in $(k_f l)^{-1}$, and Eq. (2) yields $\bar{G}(\mathbf{r},\mathbf{r}')\sim\exp[-|\mathbf{r}-\mathbf{r}'|/2l]$. At intermediate fields Eq. (2) has no simple analytic solution, nonetheless it is clear that $\bar{G}(\mathbf{r},\mathbf{r}')$ is always short ranged with decay length roughly equal to $\min\{r_c,l\}$.

Mesoscopic quantum interference effects always arise from long-ranged contributions to the average of the product of two Green functions: the *diffuson* contribution, and the *Cooperon* contribution, which is suppressed at modest fields and will not be treated here. The diffuson satisfies

$$d(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')+\int d\mathbf{r}_1\, d_0(\mathbf{r},\mathbf{r}_1)d(\mathbf{r}_1,\mathbf{r}')\,, \tag{4}$$

where $d_0(\mathbf{r},\mathbf{r}')=c_i u^2\bar{G}^+(\mathbf{r},\mathbf{r}',X+\Delta X)\bar{G}^-(\mathbf{r}',\mathbf{r},X)$ and X denotes all the external parameters such as ε_f and B which may differ between measurements. Equation (4) can be solved formally in terms of the eigenfunctions $\chi_j(\mathbf{r})$ and eigenvalues $1-\xi_j^2$ of d_0 defined by

$$\int d\mathbf{r}'\, d_0(\mathbf{r},\mathbf{r}')\chi_j(\mathbf{r}')=(1-\xi_j^2)\chi_j(\mathbf{r})\,. \tag{5}$$

One finds

$$d(\mathbf{r},\mathbf{r}')=\sum_{j=1}^{\infty}\frac{\chi_j(\mathbf{r})\chi_j^*(\mathbf{r}')}{\xi_j^2}\,. \tag{6}$$

To see the diffusive behavior of d at long distances we note that $d_0\sim\bar{G}^+\bar{G}^-$ is short ranged and expand $\chi(\mathbf{r}')$ in Eq. (5) around $\mathbf{r}$ up to second order to obtain a differential equation [11],

$$\left[\frac{l_0^2}{2}\left(-i\nabla-\frac{e}{\hbar c}\Delta\mathbf{A}(\mathbf{r})\right)^2+C(\Delta E)\right]\chi_j(\mathbf{r})=(1-\xi_j^2)\chi_j(\mathbf{r})\,. \tag{7}$$

The constant $C(\Delta E)$ is

$$C(\Delta E)=c_i u^2\int d\mathbf{r}'\,\bar{G}^+(\mathbf{r},\mathbf{r}',E+\Delta E)\bar{G}^-(\mathbf{r}',\mathbf{r},E)\,.$$

It may be shown [12] directly from Eq. (2) that $C(0)=1$; this is a manifestation of the Ward identity [7] which guarantees that the SCBA is a number-conserving approximation. An expansion for small ΔE then yields $C(\Delta E)\approx 1-i(\Delta E\tau_0/\hbar)$ where the generalized scattering rate is $1/\tau_0\equiv 2\,\mathrm{Im}[\Sigma^+(E,B)]/\hbar=2\pi c_i u^2\bar{\rho}(\varepsilon_f,B)/\hbar$, and $\bar{\rho}(\varepsilon_f,B)$ is the average density of states. The other constant in Eq. (7) has dimensions of length squared, with

$$l_0^2=\frac{c_i u^2}{2}\int d\mathbf{r}'\,|\mathbf{r}-\mathbf{r}'|^2|\bar{G}^+(|\mathbf{r}-\mathbf{r}'|)|^2\,, \tag{8}$$

i.e., the length l_0 is just the spatial range of the average Green function. From the explicit form of $\bar{G}$ in the limit $r_c\ll l$ it is easily found that $l_0^2\to 2r_c^2$ and $\tau_0\to\hbar/v$ at the center of LL, whereas when $l\ll r_c$ one obtains the familiar result $l_0^2\to l^2=2D\tau$, $\tau_0\to\tau$. Hence we define the field-dependent diffusion constant $D_0(B)\equiv l_0^2/2\tau_0$. If one sets $\Delta E=\Delta\mathbf{A}=0$ in Eq. (7) and Fourier transforms, one finds $\xi_j^2=D_0\tau_0 q^2$, and Eq. (6) has the familiar diffusion pole.

The basic statistical properties of the conductance fluctuations are determined by the correlation function

$$F(\Delta E,\Delta B)\equiv\langle\delta g(\varepsilon_f+\Delta E,B+\Delta B)\delta g(\varepsilon_f,B)\rangle\,, \tag{9}$$

where $\delta g(\varepsilon_f,B)=g(\varepsilon_f,B)-\bar{g}(\varepsilon_f,B)$. $F(0,0)=\langle\delta g^2\rangle$ gives $\mathrm{Var}(g)$, and the decay width of $F(\Delta E,0)$ and $F(0,\Delta B)$ gives the correlation ranges E_c and B_c of the fluctuations with ε_f and B. In the present work we restrict ourselves to considering two-probe conductance g, which is roughly speaking the inverse of the sum of the longitudinal and Hall resistances. For a given impurity configuration at $T=0$ this quantity may be written in operator form [13] as

$$g(\varepsilon_f)=-\frac{e^2\hbar}{4\pi L_x^2}\mathrm{Tr}\{v_x\Delta G v_x\Delta G\}\,, \tag{10}$$

where v_x is the velocity operator, and $\Delta G(\varepsilon_f)=G^+(\varepsilon_f)-G^-(\varepsilon_f)$ contains the dependence on the impurity potential. We need to calculate the average of the product of two such factors evaluated in general at slightly different values of ε_f and B. As in the conventional theory of UCF, this average can be represented diagrammatically by two conductance bubbles connected by impurity lines [3], where the largest contributions come from diagrams which behave at small momentum as $\sim(D_0\tau_0 q^2)^{-2}$.

Denote $\mathrm{Tr}\{G^a v_x G^a v_x\}$ by g^{ab}, $a,b=+,-$; a significant technical complication arises because terms of the type $\overline{g^{aa}g^{bc}}$ and $\overline{g^{aa}g^{bb}}$ are not negligible as they are in the standard theory [4,14]. It is hence convenient to simplify the terms involving g^{aa} before averaging over impurities.

Using the operator identities $v_x = (i/\hbar)[H,x]$, $G^{\pm}H = I$, $[x,v_x] = i\hbar/m$ one finds [14]

$$g^{aa} = \mathrm{Tr}\{G^a v_x G^a v_x\} = -(1/m)\mathrm{Tr}\{G^a\}.$$

Therefore each occurrence of g^{aa} may be replaced by a single Green function without velocity vertices (indicated by straight lines with a single dot in Fig. 1). Within the SCBA the leading contributions to $F(\Delta E,\Delta B)$ come from the diagrams shown in Fig. 1. In the general case the vertices need to be corrected by nondivergent ladders of type $\overline{G^a G^a}$. In the low-field limit diagrams 1(c) and 1(d) are of lower order in $(k_f l)^{-1}$ and negligible, but in the high-field limit they are of order $N^{-1}\hbar\omega_c/v$ and they are needed to cancel nonuniversal contributions of the same order from diagrams of the type 1(b).

The particular form of $\overline{G}$ only affects the diffusion pole through the constant $l_0^2(B) = 2D_0\tau_0$. We now indicate how the dependence on the degree of disorder cancels in Var(g) at arbitrary field by considering the simplest case of diagram 1(a). Following Ref. [3] we may treat the current vertex $J(\mathbf{r},\mathbf{r}')$ which connects the two diffusons in Fig. 1(a) as short ranged since it only involves products of $\overline{G}$, $J(\mathbf{r},\mathbf{r}') \approx J\delta(\mathbf{r}-\mathbf{r}')$. The identification of the points $\mathbf{r},\mathbf{r}'$ then turns the integration over the two diffusons into a trace, $\mathrm{Tr}\{d(X+\Delta X)d^{\dagger}(X+\Delta X)\} = \sum_j |\xi|_j^{-4}$. Diagram 1(a) then yields

$$F_a = 4\left(\frac{e^2\hbar}{4\pi L_x^2}\right)^2 J^2 \sum_j \frac{1}{|\xi_j|^4}, \qquad (11)$$

where at $\Delta E,\Delta B = 0$ we have $\xi_j^2 \sim D_0\tau_0 q^2$, with

$$J = \frac{c_i u^2}{A}\mathrm{Tr}\{\overline{G}^+ v_x \overline{G}^+ \overline{G}^- v_x \overline{G}^-\}. \qquad (12)$$

Although this trace involves the *impurity-averaged* Green function (not G for a given sample), within the SCBA we still have $[E-\Sigma^{\pm}(E)-H_0]\overline{G} \equiv (Z_0^{\pm})\overline{G} = 1$, and $v_x = (-i/\hbar)[Z_0^{\pm},x]$. Using this identity with appropriate choice of $Z_0^{\pm}$ in Eq. (12) yields

$$J = -\frac{c_i u^2}{A\hbar^2}\mathrm{Tr}\{(x\overline{G}^+ - \overline{G}^+ x)(x\overline{G}^- - \overline{G}^- x)\}$$
$$= l_0^2/\hbar^2 = 2D_0\tau_0/\hbar^2, \qquad (13)$$

where we have used the definition of l_0^2 in Eq. (8).

The summation of diagrams 1(b)-1(d) gives another vertex constant which after lengthy algebra is also found to be proportional to l_0^2 [12]. Adding up all diagrams with correct counting factors we find the full $T=0$ correlation function,

$$F(\Delta E,\Delta B) = \left(\frac{e^2}{\hbar}\right)^2 \frac{l_0^4}{L_x^4}\sum_j \left[\frac{1}{|\xi_j|^4} + \frac{1}{2}\mathrm{Re}\left\{\frac{1}{\xi_j^4}\right\}\right], \qquad (14)$$

where the dependence on $\Delta E,\Delta B$ comes through the dependence of the ξ_j^2 of Eq. (7) on these quantities. Equation (14) is valid at arbitrary field (except for fields near $B=0$ where it is straightforward to include the Cooperon contribution) as long as the SCBA is a reasonable approximation. This expression is *identical* to the conventional theory which neglects Landau quantization effects except for the appearance of the generalized field-dependent scattering coefficients l_0^2 and τ_0. To evaluate Var(g) one sets $\Delta E = \Delta B = 0$, in which case, as noted above, one has $\xi_j^4 = (D_0\tau_0 q^2)^2 = (l_0 q)^4/4$; *thus quite generally we see that the vertex constant $J \sim l_0^2$ cancels the diffusion pole, eliminating the dependence of Var(g) on the diffusion constant.* The standard analysis [3,4] then shows that Var$(g) = C(e^2/h)^2$, where the constant C is independent of magnetic field or sample size. Previous microscopic calculations had not made clear the origin of this cancellation and its generality; a similar cancellation occurs when evaluating the Coulomb correction to the average conductance in high magnetic field [9,12].

The correlation ranges are obtained from the decay of Eq. (14) as a function of ΔB and ΔE [3]. For $l_{\mathrm{in}} \gg L_x$ one finds $E_c = \hbar D_0(B)/L_x^2$, the standard result with $D \to D_0(B)$; however, for $l_{\mathrm{in}} \ll L_x$ one finds (neglecting energy averaging) $E_c \sim \hbar D_0(B)/l_{\mathrm{in}}^2 = \hbar/\tau_{\mathrm{in}}$ and the B dependence of the diffusion constant cancels out of E_c (of course τ_{in} may still have B dependence). On the other hand, we find that the magnetic field correlation length satisfies $B_c \sim (hc/e)/l_{\mathrm{in}}^2$. At high fields, if τ_{in} is only weakly field dependent, this implies $B_c \sim 1/D_0(B) \sim 1/r_c^2 v \sim B^{3/2}$; i.e., B_c increases with B as observed experimentally [1,2]. A detailed comparison to the data of Ref. [2] is given in Fig. 2 using the approximation that τ_{in} is independent of B, but calculating $D_0(B)$ exactly from Eqs. (3) and (8).

Although this theory contains the conventional theory of UCF as a special case, it has significant limitations

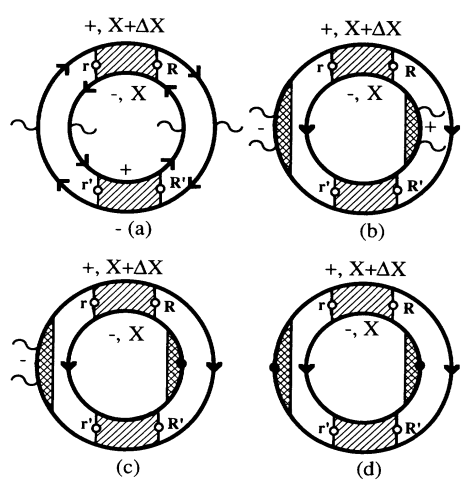

FIG. 1. Diagrams contributing to $F(\Delta E,\Delta B)$. Shaded areas denote diffusions; cross-hatched areas denote vertex corrections which arise from nondivergent ladder insertions.

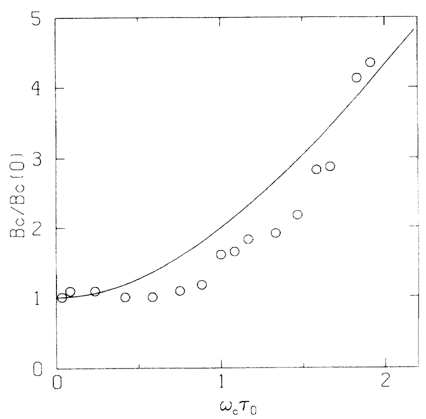

FIG. 2. Comparison of the predicted field dependence of the correlation field, $B_c(B)$ (normalized to its low-field value), to data of Ref. [2] using parameters cited therein (no free parameters). The range of $\omega_c\tau_0$ shown corresponds to the interval 0–12 T (sample is low mobility and not in the quantum Hall regime).

when applied to a two-dimensional electron gas in high magnetic field, since the theory neglects localization effects completely. These effects are suppressed by the B field when $l \ll r_c$ and often can be negligible; whereas in high field the relevant perturbative parameter is $N^{-1} \sim \hbar\omega_c/\varepsilon_f$, which is an increasing function of B for fixed density. Hence the range of system size for which the expression is approximately valid gets smaller with increasing B and the localization effects lead to the suppression of fluctuations in the Hall plateaus. This implies that the ergodic hypothesis [3] fails badly and varying B is not equivalent to changing the impurity configuration at fixed B. In the infinite system at $T=0$ the width in B of the extended states would go to zero and so would the width of the resistance steps; however, in mesoscopic systems at nonzero T these widths remain substantial (≈ 0.5 T in Ref. [1]) and our theory is relevant. Reference [1] finds $B_c \approx 0.04$ T, so B_c is roughly 8% of the step width making it difficult to obtain a statistically meaningful measure of Var(g) by varying B. Tests of the theory in the quantum Hall regime should ideally be made in a small range of B field near the center of the LL, varying some other parameters to obtain statistically independent conductance measurements. Our prediction is that for samples with short-ranged disorder the amplitude of the fluctuations will be independent of the LL index (the behavior for smooth disorder is not yet known).

The theory presented here highlights further the universality of mesoscopic fluctuation phenomena. The details of the bare quantum states are unimportant as long as they are extended since at large distances the only coherent scattering (represented by the diffuson) will be described by a diffusion equation with only the diffusion constant reflecting the nature of the underlying states. Since the diffusion constant cancels in the variance of the conductance (and in other physical properties), one finds remarkably general behavior.

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