Dimensional Expansions

Carl M. Bender, Stefan Boettcher, and Lev Lipatov[a]
Department of Physics, Washington University, St. Louis, Missouri 63130
(Received 20 March 1992)

It is proposed that the dimension D be used as a perturbation parameter. The coefficients in the resulting dimensional expansion, a series in powers of D, can be obtained in both quantum-mechanical and field-theoretic models. In this paper the first three terms in the dimensional expansion for the ground-state energy density for a self-interacting ϕ^{2K} field theory are calculated. Dimensional expansions are shown to provide accurate nonperturbative analytic results.

PACS numbers: 11.10.Kk, 03.65.Ge

In this Letter we propose a new analytical method for extracting nonperturbative solutions to problems in physics. The procedure consists of expanding the solution as a perturbation series in powers of a nonperturbative parameter, namely, D, the dimension of space-time. In any perturbative approach it is essential to have an analytic solution to the unperturbed problem to be able to obtain higher-order perturbative corrections analytically. The advantage of our procedure is that the zero-dimensional problem can usually be solved in closed form. Moreover, the zero-dimensional solution already contains nonperturbative information.

As an illustration of the nature of dimensional expansions, consider the quantum-mechanical problem of a particle confined to a spherically symmetric infinite potential well in D-dimensional space:

$$V=0 \quad (r<1), \qquad V=\infty \quad (r\geq 1). \tag{1}$$

The time-independent s-wave Schrödinger equation for this particle is

$$-\psi''(r)-[(D-1)/r]\psi'(r)=E\psi(r), \tag{2}$$

where we impose the boundary conditions $\psi(0)$ finite, $\psi(1)=0$. The eigenvalue E satisfies the quantization condition $J_{-1+D/2}(\sqrt{E})=0$, which determines E as a function of D. The eigenvalue spectrum $E_n(D)$, $n=0$, $1,2,3,\ldots$, can be expressed as series in powers of the dimension D:

$$E_n(D)=\sum_{k=0}^{\infty} a_{n,k}D^k. \tag{3}$$

These series can be used to calculate the eigenvalues to great accuracy. We find that the radius of convergence of the series for $E_0(D)$ and $E_1(D)$ is 2, for $E_2(D)$ is 4, for $E_3(D)$ is 6, and so on. The radii of convergence are determined by branch-point singularities on the negative real axis in the complex-D plane. For all n, the functions $E_n(D)$ are branches of a single analytic function $E(D)$, $E(D)$ has a square-root branch point at $D=-2$ where $E_0(D)$ and $E_1(D)$ are degenerate; a cube-root branch point at $D=-4$ where $E_0(D)$, $E_1(D)$, and $E_2(D)$ are degenerate; a fourth-root branch point at $D=-6$ where $E_0(D)$, $E_1(D)$, $E_2(D)$, and $E_3(D)$ are degenerate; and so on [1,2]. In Fig. 1 we have plotted the lowest four eigenvalues as a function of real D, $-6.5\leq D\leq 2$. Note that $E_0(D)$ and $E_1(D)$ are complex conjugates for $D<-2$, and $E_2(D)$ and $E_3(D)$ are complex conjugates for $D<-6$.

Our principal objective here is to use dimensional expansions to obtain nonperturbative solutions to quantum field theories in D-dimensional Euclidean space. The advantage of such a procedure is that the unperturbed problem is a zero-dimensional quantum field theory, which is the simplest interacting quantum field theory that can be solved in closed form. In this Letter we restrict our attention to a massless self-interacting scalar ϕ^{2K} quantum field theory described by the Euclidean space Lagrangian

$$L=\tfrac{1}{2}(\partial\phi)^2+g\phi^{2K}. \tag{4}$$

For simplicity of presentation we limit our discussion to

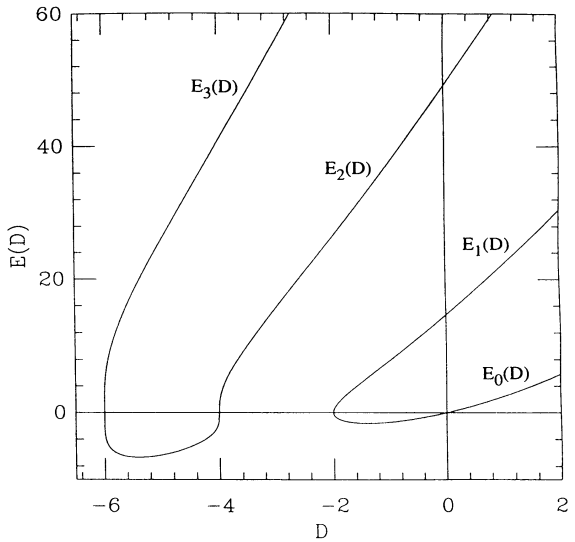

FIG. 1. Eigenvalues $E_n(D)$, $n=0,1,2,3$, of the Schrödinger problem in (2) plotted for $-6.4<D<2$.

the problem of obtaining the dimensional expansion of just one function $A_K(D)$ characteristic of the quantum field theory, where $A_K(D)$ is the logarithmic derivative of the vacuum energy density as defined below. The vacuum energy density F is defined by a functional integral

$$e^{-FV}=\int D\phi \exp\left(-\int d^D x\, L\right), \tag{5}$$

where V is the volume of D-dimensional Euclidean space and the measure is properly normalized. In terms of F we define

$$A_K(D)\equiv 2Kg^{1-D/[2K-D(K-1)]}\frac{dF}{dg}. \tag{6}$$

Note that $A_K(D)$ is independent of an additive constant in the ground-state energy due to the choice of the measure of the functional integral in (5). Note also that the definition of $A_K(D)$ contains the appropriate power of g necessary to make $A_K(D)$ dimensionless. The dimension expansion of $A_K(D)$ has the form

$$A_K(D)=\alpha+\beta D+\gamma D^2+\cdots. \tag{7}$$

We describe below several ways to calculate the coefficients $\alpha,\beta,\gamma,\ldots$.

The normalization in (6) is chosen so that the coefficient α in (7), determined by evaluating the zero-dimensional version of the functional integral in (5), is unity:

$$\begin{aligned}\alpha=A_K(0)&=2Kg\frac{dF}{dg}\\&=-2Kg\frac{d}{dg}\ln\left(\int_{-\infty}^{\infty}dx\,e^{-gx^{2K}}\right)\\&=1.\end{aligned} \tag{8}$$

The higher-order coefficients in the dimensional expansion in (7) are more difficult to calculate. However, for the special case of a free field theory ($K=1$) we can obtain the entire dimensional expansion by solving for $A_1(D)$ exactly. The functional integral in (5) is easy to evaluate when $K=1$ because it is Gaussian.

We obtain for the free energy

$$F=\frac{1}{2}\int\frac{d^Dp}{(2\pi)^D}\ln\left(1+\frac{2g}{p^2}\right)=\frac{1}{D}\left(\frac{g}{2\pi}\right)^{D/2}\Gamma(1-D/2), \tag{9}$$

where we have omitted an infinite additive constant (independent of g) associated with the zero-point energy field fluctuations. From (9) we obtain

$$A_1(D)=\left(\frac{1}{2\pi}\right)^{D/2}\Gamma\left(1-\frac{D}{2}\right)=1-\frac{D}{2}[\ln(2\pi)-\gamma]+\frac{D^2}{48}[6\ln^2(2\pi)-12\gamma\ln(2\pi)+\pi^2+6\gamma^2]+\cdots. \tag{10}$$

Note that the radius of convergence of this expansion is 2.

To test the convergence of this series numerically we evaluate the sum at $D=1$ (the case of quantum mechanics) for which the exact answer is $A_1(1)=2^{-1/2}=0.7071068\ldots$. Of course, the Taylor series in (10) converges at $D=1$. The sum of the first eleven terms in the Taylor series is $0.70695\ldots$. However, we can sum the series in (10) more efficiently if we form the diagonal Padé approximants $P_n^n(D)$. We obtain

$$P_1^1(1)=0.6159728,\quad P_2^2(1)=0.6930203,$$

$$P_3^3(1)=0.7076109,\quad P_4^4(1)=0.7071068,$$

$$P_5^5(1)=0.7071074,$$

and so on, which converges quite rapidly to the exact result.

For the interacting theory ($K>1$) it is possible to find the exact value of the coefficient β in (7) using analytical means. To do so we add and subtract a mass term to L in (4):

$$L=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2+g\phi^{2K}-\tfrac{1}{2}m^2\phi^2, \tag{11}$$

where we choose $M=g^{1/[2K-D(K-1)]}$.

The Feynman rules for the weak-coupling expansion of the Lagrangian in (11) are $1/p^2+m^2$ for a line, $-(2K)!g$ for a $2K$-point vertex, and m^2 for a 2-point vertex.

The free energy density F of L in (11) can be expressed as the sum of two quantities, F_1+F_2. F_1 is the ground-state energy density of the free theory described by L_0:

$$L_0=\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m^2\phi^2.$$

For this free theory the exact result for F_1 [see (9)] is

$$F_1=\frac{1}{D}\left(\frac{m^2}{4\pi}\right)^{D/2}\Gamma(1-D/2), \tag{12}$$

up to an additive constant independent of g. Note that (12) is singular at $D=0$.

The energy density F_2 is the sum of all connected vacuum graphs constructed from the Feynman rules above, except for the polygon graphs contributing to F_1. All such graphs are infrared and ultraviolet convergent for $0<D<2$. Each graph when evaluated gives a function of D which is finite at $D=0$ times m^{-D}. Thus, since F_1 in (12) is singular at $D=0$, F_2 is already higher-order correction in D compared to F_1. Hence, we can evaluate all these vacuum graphs at $D=0$. The result for F_2 is

$$\begin{aligned}F_2&=-m^D\ln\frac{\int_{-\infty}^{\infty}dx\,e^{-x^{2K}}}{\int_{-\infty}^{\infty}dx\,e^{-x^2/2}}\\&=-m^D\ln\left[\left(\frac{2}{\pi}\right)^{1/2}\Gamma\left(1+\frac{1}{2K}\right)\right].\end{aligned} \tag{13}$$

Combining F_1 and F_2 and using the definition of $A_K(D)$ in (6) we have

$$A_K(D)=1-\frac{D}{2}\ln\left[8e^{-\gamma_e}{}^{-1+1/K}\Gamma^2\left(1+\frac{1}{2K}\right)\right]+\gamma D^2+\cdots. \tag{14}$$

The first two terms in this series give fairly accurate numerical results at $D=1$. The exact values of $A_2(1)$ and $A_3(1)$ are 0.89065... and 1.02106... while the first two terms in (14) evaluated at $D=1$ give 0.597... (33% relative error) and 0.657... (36% relative error).

We do not know how to calculate analytically the coefficient γ of D^2 in the dimensional series (14). However, we now propose an approximate method for computing the higher-order terms in this series. This method is based on strong-coupling lattice techniques [3,4]. We briefly summarize the results below. First, the formula for $A_K(D)$ is

$$A_K(D)=\varepsilon^{DK/[2K-D(K-1)]}\left[1+2a^D\varepsilon\frac{d}{d\varepsilon}\left(\begin{array}{c}\text{connected strong-coupling}\\ \text{vacuum graphs}\end{array}\right)\right], \tag{15}$$

where

$$\varepsilon=[ga^{2K-D(K-1)}]^{-1/K} \tag{16}$$

is a dimensionless parameter which for fixed lattice spacing a and large coupling constant g is regarded as small. The Feynman rules for the vacuum graphs are $\partial^2\delta^D(x-y)$ for a line and $V_{2n}=a^{2n+D(n-1)}\varepsilon^n L_{2n}$ for a $2n$-point vertex, where L_{2n} are numerical constants depending on K. The first few L_{2n} are

$$L_2=\Gamma(3/2K)/\Gamma(1/2K),$$

$$L_4=\Gamma(5/2K)/\Gamma(1/2K)-3\Gamma^2(3/2K)/\Gamma^2(1/2K),$$

$$L_6=\Gamma(7/2K)/\Gamma(1/2K)-15\Gamma(3/2K)\Gamma(5/2K)/\Gamma^2(1/2K)+30\Gamma^3(3/2K)/\Gamma^3(1/2K),$$

and so on. The details of this calculation are given in a separate paper [5].

As is shown in Refs. [4,5], D-dimensional strong-coupling lattice graphs are polynomials in the parameter D having integer coefficients. Specifically, a vacuum graph having n lines is a polynomial of degree n in D. The lowest power of D in this polynomial is at least 1 and for most vacuum graphs the lowest power of D is much greater than 1. Therefore, only a small subset of all possible vacuum graphs can contribute to any given order in the dimensional expansion (14). In Table I we give the total number of nth-order vacuum graphs (those having n lines), the number of graphs contributing to order D, and the number of graphs contributing to order D^2.

Using the graphical rules we obtain from (15) a series representation for $A_K(D)$ in powers of D:

$$\begin{aligned}A_K(D)=\varepsilon^{DK/[2K-D(K-1)]}\{&1+D[-2L_2\varepsilon+2L_2^2\varepsilon^2-(6L_2^4-L_4^2/3)\varepsilon^4+(80L_2^6-6L_2^2L_4^2+L_6^2/60)\varepsilon^6+\cdots]\\&+D^2[(4L_2^2+2L_4)\varepsilon^2-(12L_2^3+6L_2L_4)\varepsilon^3+(12L_2^4+4L_2^2L_4)\varepsilon^4\\&+(60L_2^5+30L_2^3L_4-20L_2L_4^2/3-5L_4L_6/6)\varepsilon^5\\&-(180L_2^6+36L_2^4L_4-12L_2^2L_4^2-L_2L_4L_6)\varepsilon^6+\cdots]+\cdots\}.\end{aligned} \tag{17}$$

TABLE I. The number of graphs contributing in order D and in order D^2. Notice that graphs having an odd number of lines contribute in order D^2. However, the total number of $O(D^2)$ graphs is not much more than the number of $O(D)$ graphs.

Number of line n	Number of graphs in order D	Number of new graphs to consider in order D^2	Total number of graphs contributing in order D^2
1	1	0	1
2	1	1	2
3	0	2	2
4	2	1	3
5	0	4	4
6	3	2	5
7	0	9	9
8	10	7	17
9	0	37	37
10	25	37	52
11	0	176	176

We have carried out this calculation to order ε^{12} in the coefficient of D and to order ε^{11} in the coefficient of D^2.

To obtain the dimensional expansion in (14) it is necessary to extrapolate the series in (17) to its continuum limit $a \to 0$ (which is equivalent to $\varepsilon \to \infty$). This extrapolation is done by converting (17) to Padé form and then setting ε to ∞. The specific extrapolation procedure is described in Ref. [5]. With this extrapolation procedure we obtain the numerical value of the coefficient of D in (14) accurate to between 5% and 10% (depending on the value of K), and the coefficient of D^2 accurate to $\cdots$. Our final results are that

$$A_2(D) = 1 - 0.40284D + (0.23 \pm 0.01)D^2 + \cdots, \quad (18a)$$

$$A_3(D) = 1 - 0.34275D + (0.26 \pm 0.01)D^2 + \cdots. \quad (18b)$$

We can sum these series at $D = 1$ to obtain numerical predictions for $A_2(1)$ and $A_3(1)$. These predictions are accurate to 7% for $K = 2$ and 10% for $K = 3$.

We wish to acknowledge early and very profitable discussions with F. Cooper. One of us, L.L., thanks the Department of Physics at Washington University for its hospitality. S.B. thanks the U.S. Department of Education for National Need Fellowship support. We are grateful to the U.S. Department of Energy for financial support.

---

[a]Permanent address: Leningrad Nuclear Physics Institute, Gatchina, St. Petersburg 188 350, Russia.

[1] This interesting array of singularities is reminiscent of the singularity structure in the complex coupling-constant plane for the eigenvalues of the anharmonic oscillator. See C. M. Bender and T. T. Wu, Phys. Rev. Lett. **21**, 406 (1968); Phys. Rev. **184**, 1231 (1968). See also C. M. Bender, H. J. Happ, and B. Svetitsky, Phys. Rev. D **9**, 2324 (1974).

[2] The fact that the singularities in the complex-D plane occur at negative even integers recalls the work of Dunne and Halliday. They found that in negative even dimensions the quantum statistics of bosons and fermions interchange. In view of this abrupt reversal it is not surprising to find singularities at $D = -2, -4, -6, \ldots$. See G. V. Dunne and I. G. Halliday, Nucl. Phys. **B308**, 589 (1988); G. V. Dunne, J. Phys. A **22**, 1719 (1989).

[3] C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, Phys. Rev. D **19**, 1865 (1979).

[4] C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, Phys. Rev. D **23**, 2976 (1981).

[5] C. M. Bender, S. Boettcher, and L. Lipatov (to be published).