Noise-Induced Transitions between Attractors in Time Periodically Driven Systems

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The stochastic Landau equation for a periodically driven process with either two discretely degenerate attractors or a continuum of degenerate attractors is studied for small noise. We calculate analytically the probability P^{tr} for the transition between the attractors leading to phase diffusion in the continuous case. Our results are in good agreement with numerical simulations and in the discrete case also with experiments on periodically driven Rayleigh-Bénard convection. They explain the sensitive dependence of P^{tr} on the equation's parameters.

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In a finite physical system possessing more than one attractor there always exists a nonzero (though sometimes negligibly small) probability to switch between the attractors due to fluctuations. Of special interest is the case of attractors that are degenerate due to symmetry. This can happen continuously or discretely depending on the underlying symmetry. A simple example is a (real or complex) scalar quantity $A(t)$ governed by the stochastic Landau equation:

$$\partial_t A = [a(t) - g|A|^2]A + \sqrt{\epsilon}\,\eta(t)\,, \tag{1}$$

where $\eta(t)$ is Gaussian white noise with strength 1.

Note that if A is real the attractor is discretely degenerate whereas the degeneration is continuous [O(2) symmetry] for complex A (then we write $\eta=\eta'+i\eta''$ with two statistically independent noise terms). In the complex case we will be particularly confronted with phase diffusion because the change of the phase occurs completely randomly caused only by the presence of noise.

Whereas much is known about the process (1) when a (and g) is constant the situation is different for the time-dependent case such as, e.g.,

$$a(t) = \hat{\alpha} + \hat{\beta}\sin(\omega t)\,. \tag{2}$$

Note that by scaling time in units of ω^{-1} and A in units of $(\omega/g)^{0.5}$ one is left with three dimensionless parameters $\alpha\equiv\hat{\alpha}/\omega$, $\beta\equiv\hat{\beta}/\omega$, and $\tilde{\epsilon}\equiv\epsilon g/\omega^2$. So far there has been numerical and analytic work on Eq. (1) for real A [1-5] which was related to experiments on Rayleigh-Bénard convection in a fairly small container driven periodically in time with a low period ω so that the roll patterns appear and disappear periodically [6]. The change from "deterministic" (consecutive patterns strongly correlated) to "stochastic" (weakly correlated) behavior occurred rather suddenly with some critical value of α (keeping β constant) and could be described by a line $\alpha=\alpha(\beta)$ in parameter space [1,3,6]. In the context of pattern formation the real case corresponds to a (small) system with fixed boundaries and up-down symmetry whereas the complex case describes the situation of an annulus where the phase is completely degenerate.

One may also think of various physical realizations in the context of equilibrium transitions. A particularly attractive one is the (electrically or magnetically driven) splay-Fréedericksz transition in nematic liquid crystals, where one has an up-down symmetry corresponding to the real case or the bend-Fréedericksz transition where one has the O(2) symmetry of the complex case [7].

In this Letter we present an analytical small-noise approximation for the transition probability which is in good agreement with the numerical and experimental work. Our work was inspired largely by Ref. [5] where symmetric distribution functions were calculated. We exploit the fact that transitions between attractors mainly occur when A is small where the corresponding linear problem (Ornstein-Uhlenbeck process) can be solved rigorously whereas the evolution in the nonlinear regime is described deterministically. We are convinced that this concept illustrated here by a simple example in one and two dimensions will also be very useful for the treatment of more complicated time-dependent systems with more degrees of freedom.

In Ref. [3] a method to compute very efficiently the full correlation function by an eigenfunction analysis of the Kolmogorov operator is presented for the real case. The results coincide in the range of validity of our approximation.

Considering first the real case we see that in the absence of noise the deterministic solutions of Eq. (1)

$$A^{\rm det}(t,A_0,t_0) = \frac{A_0\exp[I(t_0,t)]}{[h(t_0,t)A_0^2+1]^{1/2}} \tag{3}$$

[with the initial condition $A^{\rm det}(t_0,A_0,t_0)=A_0$], where

$$I(t_0,t) = \int_{t_0}^{t} a(\tau)\,d\tau\,, \tag{4}$$

$$h(t_0,t) = 2g\int_{t_0}^{t}\exp[2I(t_0,\tau)]\,d\tau \tag{5}$$

can never cross the origin. Nevertheless, in the presence of even small noise the transition probability can become significant if A comes close enough to the origin. Note that for long times [as $h(t_0,t)$ becomes "large enough"] all trajectories (with $A_0\neq 0$) converge against the periodic attractors [5]

$$A^{\rm at}_{\pm}(t)=\pm \lim_{t_0\to -\infty}\frac{\exp[I(t_0,t)]}{[h(t_0,t)]^{1/2}} \tag{6}$$

$$=\begin{cases}\pm\{[\exp(4\pi a)-1]/h(t-T,t)\}^{1/2}, & \text{for } a>0\,,\\ 0, & \text{elsewhere}\,,\end{cases} \tag{7}$$

where $T=2\pi/\omega$ is the period of the system. Clearly in deriving Eq. (7) we employed the periodicity of $a(t)$ [compare Eq. (2)]. We see that two different attractors exist only for $a>0$ and $a=0$ is the (deterministic) stability limit of the solution $A\equiv 0$.

Let us focus on a trajectory with $A>0$ which approaches the origin as the coefficient $a(t)$ of Eq. (1) is negative. In the limit of small noise the trajectory will be "close" to one of the deterministic trajectories given by Eq. (3) if transitions across the origin have not yet occurred. It is clear that the transition probability grows as the trajectory approaches the origin so that it should be instructive to regard a trajectory which has already come sufficiently close to the origin so that the nonlinear term of Eq. (1) can be neglected. In the linear regime our problem corresponds to an Ornstein-Uhlenbeck process whose Fokker-Planck equation is solved rigorously [8] by a Gaussian probability distribution with variance

$$\sigma(t)=\epsilon\exp[2I(t_0,t)]\int_{t_0}^{t}\exp[-2I(t_0,\tau)]d\tau\,. \tag{8}$$

It is convenient to introduce the new variable

$$z=A/\sqrt{\sigma(t)}\quad\text{for } t>t_0\,. \tag{9}$$

Then the probability to find z at time t if $A(t_0)=A_0$ is given by

$$P(z,t|A_0,t_0)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(z-z_0)^2}{2}\right], \tag{10}$$

where

$$z_0=z_0(t,A_0,t_0)=\frac{A_0}{\{\epsilon\int_{t_0}^{t}\exp[-2I(t_0,\tau)]d\tau\}^{1/2}}$$
$$\simeq\left[\epsilon\int_{t_0}^{t}[A^{\rm det}(\tau,A_0,t_0)]^{-2}d\tau\right]^{-1/2}. \tag{11}$$

In the last step we have made use of the linear-range approximation of Eq. (3): $A^{\rm det}(t,A_0,t_0)\simeq A_0\exp[I(t_0,t)]$.

From Eq. (10) one obtains for the transition probability

$$P^{\rm tr}(t,A_0,t_0)=\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty}\exp\left[-\frac{y^2}{2}\right]dy$$
$$=\frac{1}{2}\,\mathrm{erfc}\left[\frac{z_0}{\sqrt{2}}\right] \tag{12}$$

[the probability that $A(t)<0$ if $A_0>0$] the (conjugate) error function [9] depending only on z_0. The integral in Eq. (11) will be completely dominated by the maximum of the integrand and thus practically independent of its limits. This becomes even more obvious in our next step of the approximation where we expand the exponent in Eq. (11) around its maximum at time t^* supposing the limits of integration t and t_0 are far enough away from t^* (method of steepest descent [10]). This yields

$$z_0^2(t,A_0,t_0)\simeq\frac{[A^{\rm det}(\tau,A_0,t_0)]^2}{\epsilon}\times\left[\frac{\partial_\tau^2\ln[A^{\rm det}(\tau,A_0,t_0)]}{\pi}\right]^{1/2}\Bigg|_{\tau=t^*}, \tag{13}$$

which depends on the behavior of the deterministic trajectory at its minimum at t^*, i.e., at the point where $a(t^*)=0$ and $\partial_t a(t^*)>0$. We note that Eq. (13) can be regarded as the small-noise approximation for any comparable stochastic process. Consistent with our approximation A_0 may even lie in the nonlinear regime.

Still the question remains which trajectories we have to include in order to obtain the averaged transition probability of the system. Here we restrict our analysis to ranges of parameters of α, β, and $\tilde\epsilon$ where "statistically relevant" trajectories converge rapidly within one period towards their attractors and can be considered to be practically on their attractors as they decrease into the linear regime. This will happen if $(A^{\rm at})^2(t^*)h(t^*-T,t^*)\gg 1$ [compare Eq. (3) [11]] which holds, as one can convince oneself, for $\alpha\gtrsim 0.4$. Then we obtain from Eq. (13)

$$z_0^2\simeq\omega\frac{(\beta^2-\alpha^2)^{1/4}}{\epsilon\sqrt{\pi}}[A^{\rm at}_{\pm}(t^*)]^2\,. \tag{14}$$

Approximating $h(t^*-T,t^*)$ in Eq. (7) by another application of the method of steepest descent finally leads to

$$[A^{\rm at}_{\pm}(t^*)]^2\simeq\omega\frac{(\beta^2-\alpha^2)^{1/4}}{2g\sqrt{\pi}}[1-\exp(-4\pi\alpha)]\times\exp\{4[\alpha\arccos(\alpha/\beta)-(\beta^2-\alpha^2)^{1/2}]\}\,. \tag{15}$$

Note that for $\epsilon\to 0$ we recover the deterministic stability limit; i.e., the transition probability stays finite only for $\alpha\to 0$ [then Eqs. (14) and (15) yield $z_0^2\propto\alpha/\tilde\epsilon$].

Figure 1 shows lines of constant z_0 in the (α,β) plane obtained with this formula together with experimental results from Ref. [6] (squares). The value $\tilde\epsilon\equiv\epsilon g/\omega^2=(0.0184)^2$ was measured independently. [Actually Fig. 1 can be used for other values of $\tilde\epsilon$ as well by noting that the quantity $z_0\tilde\epsilon^{0.5}$ is noise independent; compare Eqs. (14) and (15).] For some values of z_0 we have listed $P^{\rm tr}$ obtained from Eq. (12) in the inset of Fig. 2 and compared the result with numerical simulations.

We note that the correlation function decays like $\langle A(t+nT)A(t)\rangle=k_1^n\langle A(t)^2\rangle$ in our approximation where $k_1=1-2P^{\rm tr}$ corresponds to the eigenvalue of the Kolmogorov operator introduced in Ref. [3]. The average time

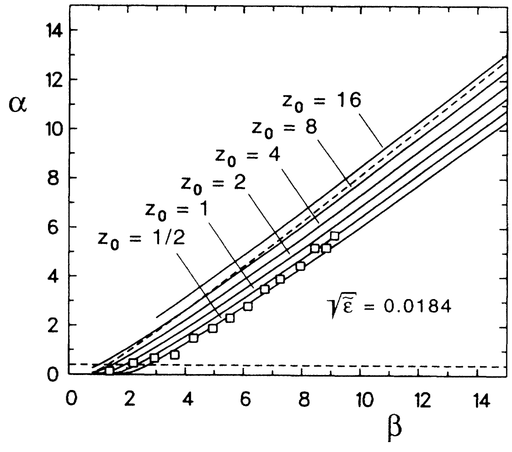

FIG. 1. Lines of constant z_0 [see Eqs. (14) and (15)] and thus also of constant transition probability P^{tr} [see Eq. (12)]. The dashed lines [$\alpha=0.4$ and $z_0^2\bar{\epsilon}(\beta^2-\alpha^2)^{-0.5}=0.007$] indicate the range of validity of our approximation which should hold accurately in the region between them [compare Eq. (18) and the discussion preceding Eq. (14)]. The squares are taken from experiments of Ref. [6]. The deviation of the theory from the experimental data for small α (below the lower dashed line) is at least partly a consequence of our approximations (compare simulations of Ref. [1]).

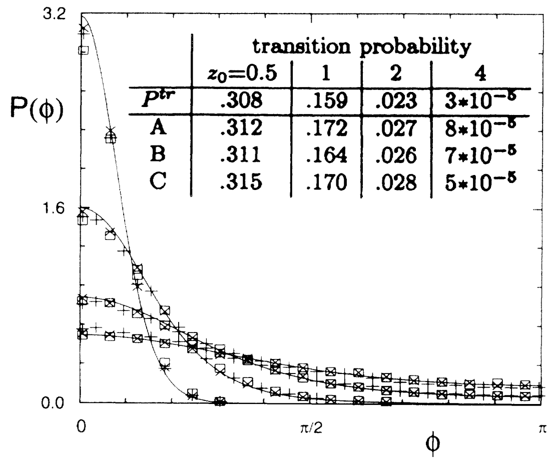

FIG. 2. Comparison of analytical results with numerical simulations. Inset: Comparison of the transition probability P^{tr} (real case) from Eq. (12) with simulations for three parameter sets (A: $\beta=2.5$, $\bar{\epsilon}^{0.5}=0.0184$; B: $\beta=5$, $\bar{\epsilon}^{0.5}=0.0184$; C: $\beta=5$, $\bar{\epsilon}^{0.5}=4\times0.0184$). The large relative error occurring at the higher values of z_0 is connected with the sensitive dependence of the transition probability on α and β. The corresponding deviation in the (α,β) plane would be very small. The curves show the probability density of the phase in the complex case obtained from Eq. (16) ($z_0=0.5,1,2,4$ as counted from below at the peaks). The symbols +, ×, and □ give the results of simulations with the parameter sets A, B, and C used before.

t_s (in units of the period $T=2\pi/\omega$) between two transitions across the origin used in Ref. [1] is generally given by $t_s=(P^{tr})^{-1}$.

In the complex case Eq. (1) corresponds to a two-dimensional problem which decouples in the linear range yielding two independent Ornstein-Uhlenbeck processes. Solving them as we did before a transformation to polar coordinates and integration over the modulus yields for the probability distribution $P(\phi)=P(\phi,t|R_0,t_0)$ of the phase with the initial condition $A(t_0)=|A(t_0)|\equiv R_0$ [i.e., $\phi(t_0)=0$]

$$P(\phi)=\int_0^\infty \frac{z}{2\pi}\exp(-\tfrac{1}{2}[z^2+z_0^2-2zz_0\cos\phi])dz$$
$$=\frac{\exp(-z_0^2/2)}{2\pi}+\frac{z_0\cos(\phi)}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}z_0^2\sin^2\phi)$$
$$\times\tfrac{1}{2}[1+\mathrm{erf}(z_0\cos\phi/\sqrt{2})]\,. \qquad (16)$$

Again we can argue that $z_0=z_0(t,R_0,t_0)$ [compare Eq. (13)] does practically not depend on t and t_0 but only on the deterministic trajectory (with initial conditions R_0,t_0) in the vicinity of its minimum. So neglecting contributions from the nonlinear range should give a good approximation provided the deterministic trajectory has its minimum well inside the linear range.

We can extrapolate the deterministic trajectory $R^{\det}$ of the modulus R into the nonlinear regime as we did in the case of real A. Since the equation of motion for R is essentially identical with Eq. (1) [12], z_0 turns out to be the same as in the real case.

In Fig. 2 we compare the result of Eq. (16) with numerical simulations. From Eq. (16) we see that due to the first term on the right-hand side phase diffusion will grow rapidly as z_0 becomes small ($z_0\sim1$) analogously to the transition probability in the real case. For larger z_0, however, the second term of Eq. (16) dominates leading to an approximately Gaussian distribution for $\sin\phi$ with a variance

$$\langle\sin^2\phi\rangle\simeq z_0^{-2}\,. \qquad (17)$$

Literally speaking, the system still makes use of its possibility to rotate the trajectory continuously in the complex plane even when the transition probability in the real case has already become negligibly small which happens very abruptly with growing z_0.

The decay factor of the correlation function is now given by $k_1=\langle\cos\phi\rangle$.

Finally we summarize the two basic assumptions necessary for the validity of our approximation: (i) strong convergence of the trajectories ($\alpha\gtrsim0.4$) and (ii) the attractors evolve well into the linear range. When these conditions are met the above analysis shows that to leading order only the interval $t^*-\Delta t<t<t^*+\Delta t$ with $\Delta t\sim1.5/(\beta^2-\alpha^2)^{0.25}$ (which is well within the linear range) contributes to the transition probability and phase diffusion. A criterion for the second condition is obtained by checking whether in the above interval the attractors are described by the linear-range approximation [13].

This means $h(t_1,t_2)[A^{\rm at}(t_1)]^2 \ll 1$ for $t_{1/2}=t^* \mp \Delta t$ [compare Eq. (3)]. Expanding the exponent in $h(t_1,t_2)$ around t^* then leads to the criterion

$$h(t_1,t_2)[A^{\rm at}(t_1)]^2 < \frac{2g[A^{\rm at}(t^*)]^2}{\omega(\beta^2-\alpha^2)^{1/4}} \int_{-1.5}^{+1.5} \exp(y^2)dy$$

$$\simeq \frac{25 z_0^2 \tilde{\epsilon}}{(\beta^2-\alpha^2)^{1/2}} \ll 1 \,. \qquad (18)$$

Small noise is required for two different reasons. First if noise is extremely strong convergence of the trajectories in the nonlinear regime may be endangered and second by making noise small enough we can ensure that the transition from stochastic to deterministic behavior will happen for parameters α and β where the attractors come well inside the linear range [compare Eq. (18)].

Note that our approximation was possible because we could show that in both the real and the complex cases the influence of noise on the trajectories increases like A^{-2} as they approach the origin [compare Eq. (11)] so that transition processes are completely dominated by the neighborhood of t^* where A has its minimum. Furthermore, the strong convergence (or focusing) of the trajectories on the attractors together with the fact that the minima of the attractors depend exponentially on α and β [compare Eq. (14)] appears to be the cause for the sensitive dependence of the transition rate on the parameters α and β which we have mentioned in the beginning of this Letter.

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[11] Treating the trajectories in the nonlinear regime deterministically can be understood as the zero-order term of a small noise expansion of the stochastic differential equation (see, e.g., Chap. 6 of [8]).

[12] Actually there is an additional term $\epsilon/2R$ resulting from the Ito formalism for coordinate transformations (see Ref. [8]) which can, however, be neglected in the nonlinear range.

[13] In principle, corrections to the linear-range part of our approximation, Eq. (10), can be derived from a systematic expansion in the nonlinearity.