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Generalized Adiabatic Invariants in One-Dimensional Hamiltonian Systems

P. Helander, M. Lisak, and V. E. Semenov[(a)]

Institute for Electromagnetic Field Theory and Plasma Physics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

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The concept of adiabatic invariance in one-dimensional Hamiltonian systems $H(p,q;\lambda)$ is generalized to include the case when the time derivative of the slowly varying parameter λ is given by $\dot{\lambda}=f(H,\lambda)\varphi(q,p)$, where f and φ are arbitrary functions, and q and p are the canonical coordinates.

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Adiabatic invariants are well known in classical Hamiltonian mechanics [1,2]. They represent quantities which are conserved to a high degree of accuracy in a system that undergoes slow changes. In 1911, A. Einstein reportedly demonstrated the adiabatic invariance of the action integral for a pendulum with a slowly varying length [2], and since then, a large number of works have been devoted to the study of adiabatic invariants; see, for example, Refs. [3–5] and the literature cited therein. Applications and generalizations have been found in many fields, such as space and fusion plasma physics [6,7], celestial mechanics [8], quantum mechanics [9], and nonlinear wave propagation [10]. In this study, we show that in one-dimensional Hamiltonian systems a generalization of the classical adiabatic invariant can be constructed, which is conserved under much more general time variations of the system parameters than what is required for the conservation of the ordinary action integral.

Consider a system described by some Hamiltonian $H(q,p;\lambda(t))$, which depends explicitly on time through the parameter $\lambda(t)$. If the system executes finite oscillations, the action integral is defined by

$$I=\frac{1}{2\pi}\int p\,dq\,, \tag{1}$$

where q and p denote the generalized coordinate and momentum, respectively. The integration in Eq. (1) is to be carried out over one period of oscillation at fixed values of λ and H; thus $p=p(q,H,\lambda)$ and $I=I(H,\lambda)$. The action is known to be an adiabatic invariant. This means that, if λ is approximately constant, $|\dot{\lambda}T/\lambda|=\varepsilon \ll 1$, where T is the period of oscillation, and the time variation of $\dot{\lambda}$ is also of order ε, then I changes little in comparison with ε during one period of oscillation:

$$\Delta I=\int_t^{t+T}\dot{I}\,dt \ll \varepsilon I\,. \tag{2}$$

In actual fact, for one-dimensional systems, the action has been shown to be adiabatically conserved with exponential accuracy [3,11], i.e., to all orders of ε, not only to the first order, as indicated in Eq. (2).

In the case when the parameter λ changes slowly, but its time derivative $\dot{\lambda}$ varies rapidly, the action is not conserved in general. However, if $\dot{\lambda}$ is related to the state of system by a function of the following form

$$\dot{\lambda}=f(H,\lambda)\varphi(q,p)\,, \tag{3}$$

we have found that a generalized adiabatic invariant J can be constructed,

$$J=\frac{1}{2\pi}\int \psi(q,p)\,dq\,, \tag{4}$$

where $\partial\psi/\partial p=\varphi(q,p)$. In order to demonstrate that this indeed is an invariant, we show that its average time variation vanishes. The time derivative of J is

$$\dot{J}=\left(\frac{\partial J}{\partial H}\frac{\partial H}{\partial \lambda}+\frac{\partial J}{\partial \lambda}\right)\dot{\lambda}$$
$$=\frac{1}{2\pi}\left(\frac{\partial H}{\partial \lambda}\int\varphi\frac{\partial p}{\partial H}dq+\int\varphi\frac{\partial p}{\partial \lambda}dq\right)f(H,\lambda)\varphi(q,p)\,. \tag{5}$$

Averaging this expression over the period of one oscillation, and using the equation of motion $\dot{q} = \partial H/\partial p$, we obtain

$$\langle \dot{J} \rangle = \frac{1}{2\pi T}\left[\int \varphi f(H,\lambda)\frac{\partial H/\partial\lambda}{\partial H/\partial p}dq\int\frac{\varphi\,dq}{\partial H/\partial p} + \int\varphi\frac{\partial p}{\partial\lambda}dq\int\varphi f(H,\lambda)\frac{\varphi\,dq}{\partial H/\partial p}\right], \quad (6)$$

where the brackets $\langle\,\rangle$ denote the averaging. Because of the slow variation of λ and H, $f(H,\lambda)$ need not be averaged in Eq. (6), and the averaging can be taken over the motion which would occur if λ remained constant. Moreover, since

$$\left(\frac{\partial H}{\partial\lambda}\right)_{q,H} = \frac{\partial H}{\partial p}\frac{\partial p}{\partial\lambda} + \frac{\partial H}{\partial\lambda} = 0, \quad (7)$$

Eq. (6) shows that, to the requisite accuracy, we have $\langle\dot{J}\rangle = 0$, and thus J is a first-order adiabatic invariant. The invariant J is a generalization of the ordinary adiabatic invariant I. In the case when the function φ is constant so that λ varies steadily, I and J are equal; if φ is not constant, I and J are in general different. The parameter λ is then subjected to small but rapid fluctuations, and the adiabatic invariance of I breaks down, as noted, e.g., in Ref. [12]. The generalized adiabatic invariant, J, however, remains constant.

The invariant J can be written as

$$J = -\int\int\varphi\,dq\,dp\,. \quad (8)$$

This shows that J is invariant under canonical transformations, since in going over to new canonical coordinates (Q,P), the area is preserved,

$$\frac{\partial(Q,P)}{\partial(q,p)} = 1\,, \quad (9)$$

which implies that $dq\,dp$ can be replaced by $dQ\,dP$ in Eq. (8).

The customary way of introducing the action variable in classical mechanics is by carrying out a canonical transformation using the abbreviated action

$$S_0(q,I,\lambda) = \int^q p\,dq \quad (10)$$

as a generating function, see, e.g., Refs. [1,2]. The generalized adiabatic invariant can be obtained in a similar manner, but canonical transformations cannot be used. Instead, we shall consider more general transformations $Q = Q(q,p,t)$, $P = P(q,p,t)$ satisfying

$$\frac{\partial(Q,P)}{\partial(q,p)} = \varphi(q,p)\,. \quad (11)$$

Quite analogously to the case of ordinary canonical transformations, this implies the existence of generating functions $F_1(q,Q,t)$ and $F_2(q,P,t)$ such that

$$\begin{aligned} \psi &= \left(\frac{\partial F_1}{\partial q}\right)_{Q,t} = \left(\frac{\partial F_2}{\partial q}\right)_{P,t}, \\ P &= -\left(\frac{\partial F_1}{\partial Q}\right)_{q,t}, \\ Q &= \left(\frac{\partial F_2}{\partial P}\right)_{q,t}, \end{aligned} \quad (12)$$

where again $\partial\psi/\partial p = \varphi(q,p)$. From the equations of motion and Eq. (11), we find the time derivatives of the new phase space coordinates,

$$\begin{aligned} \dot{Q} &= \varphi\left(\frac{\partial H}{\partial P}\right)_{Q,t} + \left(\frac{\partial Q}{\partial t}\right)_{q,p}, \\ \dot{P} &= -\varphi\left(\frac{\partial H}{\partial Q}\right)_{P,t} - \left(\frac{\partial P}{\partial t}\right)_{q,p}, \end{aligned} \quad (13)$$

which, by using Eq. (12), can be transformed to the following equations of motion:

$$\begin{aligned} \dot{Q} &= \varphi\left(\frac{\partial H}{\partial P}\right)_{Q,t} + \left(\frac{\partial}{\partial P}\right)_{Q,t}\left(\frac{\partial F_2}{\partial t}\right)_{q,P}, \\ \dot{P} &= -\varphi\left(\frac{\partial H}{\partial Q}\right)_{P,t} - \left(\frac{\partial}{\partial Q}\right)_{P,t}\left(\frac{\partial F_2}{\partial t}\right)_{q,P}. \end{aligned} \quad (14)$$

Apart from the extra factors φ, these coincide with the usual, canonically transformed equations of motion.

Armed with these results, we can now introduce "generalized action-angle variables" (ϑ,J) by carrying out a noncanonical transformation with the generating function

$$S(q,J,\lambda) = F_2(q,J,\lambda) = \int^q \psi\,dq\,, \quad (15)$$

where J is defined as before, Eq. (4). The Hamiltonian then becomes independent of the angle variable ϑ, and the equations of motion (14) read

$$\begin{aligned} \dot{\vartheta} &= \varphi\frac{\partial H}{\partial J} + \dot{\lambda}\frac{\partial}{\partial J}\frac{\partial S}{\partial\lambda}, \\ \dot{J} &= -\dot{\lambda}\frac{\partial}{\partial\vartheta}\frac{\partial S}{\partial\lambda}. \end{aligned} \quad (16)$$

To the first order in ε, we can neglect the second term in the equation for $\dot{\vartheta}$, and consider the generalized action J as a constant on the right-hand sides of Eqs. (16). If $\dot{\lambda}$ is as in Eq. (3), the variation of J over one period then vanishes:

$$\Delta J = \int_t^{t+T}\dot{J}\,dt = -\int_{\vartheta(t)}^{\vartheta(t+T)}\frac{\partial}{\partial\vartheta}\left(\frac{\partial S}{\partial\lambda}\right)\frac{\dot{\lambda}\,d\vartheta}{\varphi\,\partial H/\partial J} = 0\,, \quad (17)$$

which again proves that J is a first-order adiabatic invariant. This result can be strengthened by introducing a new time variable τ by $d\tau/dt = \varphi$ and writing $f(H,\lambda)$

$=\varepsilon g(J,\lambda)$. We then have from Eqs. (3) and (16)

$$\frac{d\lambda}{d\tau}=\varepsilon g(J,\lambda)\,,$$
$$\frac{d\vartheta}{d\tau}=\frac{\partial H}{\partial J}+\frac{d\lambda}{d\tau}\frac{\partial}{\partial J}\frac{\partial S}{\partial \lambda}\,, \quad (18)$$
$$\frac{dJ}{d\tau}=-\frac{d\lambda}{d\tau}\frac{\partial}{\partial \vartheta}\frac{\partial S}{\partial \lambda}\,.$$

If now g only depends on λ and not on J, we can solve the first of these three equations and obtain $\lambda(\varepsilon\tau)$. The remaining system then coincides with the usual system of equations for action-angle variables in a system with a slowly varying parameter, which implies that J is conserved with exponential accuracy [11].

As a simple example of a generalized adiabatic invariant, we can consider the following harmonic oscillator with a slowly varying, but rapidly fluctuating frequency ω:

$$H=\tfrac{1}{2}p^2+\tfrac{1}{2}\omega^2(t)q^2\,,$$
$$\dot{\omega}=\delta(\omega)(aq^{2m}+bp^{2n})\,. \quad (19)$$

Here a and b are constants, and $\delta(\omega)$ denotes any function of the frequency ω. Note that $\omega(t)$ is to be considered as an explicit function of time. The generalized action J becomes

$$J=\frac{2}{\pi}\int_0^{\sqrt{2H}/\omega}\left[aq^{2m}+b\frac{(2H-\omega^2q^2)^n}{2n+1}\right]\times(2H-\omega^2q^2)^{1/2}dq$$
$$=a\frac{(2m-1)!!}{(2m+2)!!}\frac{(2H)^{m+1}}{\omega^{2m+1}}+b\frac{(2n-1)!!}{(2n+2)!!}\frac{(2H)^{n+1}}{\omega} \quad (20)$$

and represents a conserved quantity, provided $\delta(\omega)$ is sufficiently small.

As a second example of the use of the generalized invariant, we consider the propagation of a weakly damped plane wave in a medium where the permittivity ε and the conductivity σ depend on the wave amplitude. The Helmholtz equation reads

$$\frac{d^2E}{dx^2}+k^2[\varepsilon(|E|)-i\sigma(|E|)]E=0\,, \quad (21)$$

where E is the electric field, x the coordinate in the direction of propagation, and k the wave vector. Introducing q, ϕ, and λ by

$$E=q(x)e^{-ik\phi(x)}\,,$$
$$\lambda=q^2\frac{d\phi}{dx}\,, \quad (22)$$

we can rewrite the imaginary part of Eq. (21) as

$$\dot{\lambda}=-kq^2\sigma(q)\,, \quad (23)$$

which shows that λ varies slowly if the damping is weak. The real part of the Helmholtz equation corresponds to the Hamiltonian

$$H=\frac{k^2p^2}{2}+\frac{\lambda^2}{2q^2}+V(q)\,, \quad (24)$$

where x plays the role of time, $p=\dot{q}/k^2$, and $dV/dq=q\varepsilon(q)$. Thus, the Helmholtz equation (21) has been reduced to a one-dimensional Hamiltonian system with a slowly varying parameter, and we can immediately write down the generalized invariant

$$J=-\frac{1}{\pi}\int_{q_{\min}}^{q_{\max}}\sigma(q)q^2\left[2H-2V(q)-\frac{\lambda^2}{q^2}\right]^{1/2}dq\,, \quad (25)$$

where $q_{\min}$ and $q_{\max}$ are the roots to the equation

$$H=V(q)+\lambda^2/2q^2\,. \quad (26)$$

Since J represents a conserved quantity, the solution to the Helmholtz equation is now easily obtained. In the linear case, for instance, when ε and σ are independent of the amplitude q, the constancy of J implies that $H_0^2=H^2-\varepsilon\lambda^2$ is constant. From the relation

$$\dot{H}=\frac{dH}{d\lambda}\dot{\lambda}=-k\sigma\left[\frac{H^2-H_0^2}{\varepsilon}\right]^{1/2} \quad (27)$$

and Eq. (24), we then find

$$H=H_0\cosh(k\sigma x/\varepsilon^{1/2})\,, \quad (28)$$

$$q^2=(1/\varepsilon)(H+H_0\cos 2k\varepsilon^{1/2}x)\,. \quad (29)$$

In the nonlinear case, the solution, though more complicated, is similarly conveniently obtained by using the invariance of J.

In summary, we have constructed a generalization of the classical adiabatic invariant in one-dimensional Hamiltonian systems, which is conserved under more general conditions than the classical invariant. In view of the universal applicability of adiabatic invariants in physics, the generalized invariant can be expected to be useful for a variety of physical problems.

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[a]Permanent address: U.S.S.R. Academy of Sciences, 46 Ulyanov Street, Nizhny Novgorod 603600, Russia.

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