Spectral Density Singularities, Level Statistics, and Localization in a Sparse Random Matrix Ensemble

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We study the eigensolution statistics of large $N \times N$ real and symmetric sparse random matrices as a function of the average number p of nonzero matrix elements per row. In the very sparse matrix limit (small p) the averaged density of states deviates from the Wigner semicircle law with the appearance of a singularity $\langle \rho(E) \rangle \propto 1/|E|$ as $E \to 0$. A localization threshold is identified at $p_q \simeq 1.4$ via a simple criterion based on the density fluctuations, and the nearest-level-spacing function $P(S)$ is shown to obey the Wigner surmise law in the delocalized phase ($p > p_q$). Our findings are in agreement with previous supersymmetric and replica theories and studies of the Anderson transition in dilute Bethe lattices.

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Many problems of both classical and quantum physics have discrete representations in terms of $N \times N$ random matrices. From the complete eigensolutions for every member of a corresponding random matrix ensemble, averages and fluctuations of interesting physical quantities can be computed. The continuum limit must be recovered when the order N of the matrix is large. The first kind of matrices, where randomness plays an important role, was introduced a long time ago in the context of nuclear physics by Wigner and Dyson [1-4]. In the case of real and symmetric matrices the exactly solvable Gaussian orthogonal ensemble (GOE) [3] is defined, where the averaged density of states (DOS) $\langle \rho(E) \rangle$ obeys a simple semicircle law [4]. The corresponding eigensolution fluctuations can be studied via the nearest-level-spacing distribution function $P(S)$, which is known to follow the Wigner surmise [4], a universal form which depends only on symmetry and implies strongly correlated eigenvalues repelling their closest neighbors.

The present work is motivated by the fundamental problem of electronic structure in disordered lattices in connection with Anderson localization [5], where the matrix representation is known as the tight-binding random matrix ensemble (TBRME) [6]. In the absence of spin effects the TBRME consists of real and symmetric sparse random matrices being drastically different from the GOE. Only in the mean-field limit of $d = \infty$, when each site is allowed to extend its hopping range to all other sites, do the $N \times N$ matrices become full corresponding to the GOE. A sparse random matrix ensemble (SRME) can be also obtained by diluting the GOE matrices and is common in a variety of problems ranging from dilute spin systems [7] to combinatorial optimization [8]. Such a SRME [9-11] is characterized by a finite mean number p of randomly placed nonzero elements per matrix row and allows one to consider the limits of validity of the Wigner-Dyson theories and their universal statistics, when the matrices deviate strongly from the GOE. It can be used as a model for studying the quantum-mechanical behavior of systems that are classically chaotic [12], and also fluctuation properties of disordered conductors [13]. More important is the fact that the SRME permits the appearance of a delocalization-localization Anderson transition for a critical value of $p = p_q$. Finally, an extra reason for considering this model is that it is amenable, to some extent, to various analytical treatments using the replica and supersymmetric methods [9-11], unlike the original TBRME on two- and three-dimensional lattices.

We consider real and symmetric $N \times N$ matrices

$$\mathbf{H} = \sum_{i,j=1}^{N} H_{i,j} |i\rangle\langle j| \, , \tag{1}$$

written in a convenient orthogonalized basis set ($|i\rangle$, $i = 1, 2, \ldots, N$). The matrix elements $H_{i,j}$ ($= H_{i,j}^* = H_{j,i}$) are independent identically distributed random variables chosen from the probability distribution

$$P(H_{i,j}) = \left(1 - \frac{p}{N}\right) \delta(H_{i,j}) + \frac{p}{2N} \left[\delta\left(H_{i,j} - \frac{1}{\sqrt{p}}\right) + \delta\left(H_{i,j} + \frac{1}{\sqrt{p}}\right) \right] . \tag{2}$$

The normalization condition $\langle \mathrm{Tr}\mathbf{H}^2 \rangle = N$ sets the energy scale and guarantees that the DOS has a compact structure in the region $[-2, 2]$, when $p = N$. For $p < N$ the majority of eigenvalues still lie in $[-2, 2]$ but exponential tails develop outside this region. When $p = N$ the GOE limit is obtained and small-p values should simulate more realistic dilute-lattice situations. Extensions of the model are obtained if we vary the ratio between the positive and negative matrix elements, or introduce a continuous, rather than binary, distribution. No qualitative changes in the results reported here were found for such extended SRMEs.

Since the known remarkable analytical solutions for the SRME [9-11] are, nevertheless, limited in their extent we chose to study the problem numerically. The method of calculation relies on the numerical computation of eigenvalues and eigenvectors in finite samples from the SRME. Our results are based on allowing the matrix size N to vary, for a given p, in order to determine the large-N behavior. The matrix ensemble we used consists of matrices of sizes up to 2000×2000. We are aiming, on one hand, for the independent confirmation of the previous analytical results, and since our approach does not suffer from the same sorts of limitations, we can answer an even more general set of questions.

We first focus, in Fig. 1, on the averaged DOS and particularly demonstrate the appearance of the $\langle\rho(E)\rangle \propto 1/|E|$ singularity as $|E| \to 0$ in the very sparse matrix limit $p \ll N$. By significantly lowering p we observe two characteristic features of the corresponding DOS: First, for special energies whose number progressively increases, δ-function peaks are seen. They are special degenerate states due to the dilute structure of the matrices and they were explained in [14] for quantum percolation models. The second and more important observation refers to the continuous component of the spectrum close to the center. For small-p values a singularity of the form of Dyson's equation [15],

$$\langle\rho(E)\rangle \propto 1/|E|\,|\ln(|E|)|^3, \text{ as } |E| \to 0\,, \tag{3}$$

appears. Since the E values for the diverging $\langle\rho(E)\rangle$ are very close to zero and the DOS is not visible due to the coarse-graining procedure involved we choose to plot the averaged integrated density of states (IDOS) against E by including a range of very small energies. In the double logarithmic plot of Fig. 2 the law of Eq. (3) implies that for small-p values the data should be straight lines. Despite the various sources of error the data lie rather accurately on straight lines and the peak of the form $\langle\rho(E)\rangle \propto 1/|E|$ is clearly displayed. From the results of [9-11] a power of 2 instead of 3 is expected for the logarithmic part of the singularity. This deviation is of no significance and, if not genuine, could also be understood as arising from numerical difficulties in our approach due to the very small energy values considered. Such $1/E$ singular structure for the DOS was previously shown [16] in disordered lattices with the randomness in the off-diagonal matrix elements further related to log-normal distributions, $1/f$ noise phenomena, and localization. At values of p close to p_c we see a dip on the DOS near the band center, as in [14].

Next, we turn to the DOS fluctuations which are estimated from the variance, $\langle[\delta n(E)]^2\rangle = \langle[n(E)]^2\rangle - \langle n(E)\rangle^2$, of the number of eigenvalues $n(E)$ in an energy bin of width E. From ordinary Poisson statistics the variance should be proportional to the mean $\langle n(E)\rangle$. However, this result refers only to localized states. As a consequence of the validity of the Wigner-Dyson statistics [1-4] $\langle[\delta n(E)]^2\rangle$ is much lower for delocalized states and varies only logarithmically with $\langle n(E)\rangle$ [17]. Therefore, the relative variance $\langle[\delta n(E)]^2\rangle/\langle n(E)\rangle$ must be of order 1 when the states in the energy bin are localized and much smaller than 1, decreasing with $\langle n(E)\rangle$,

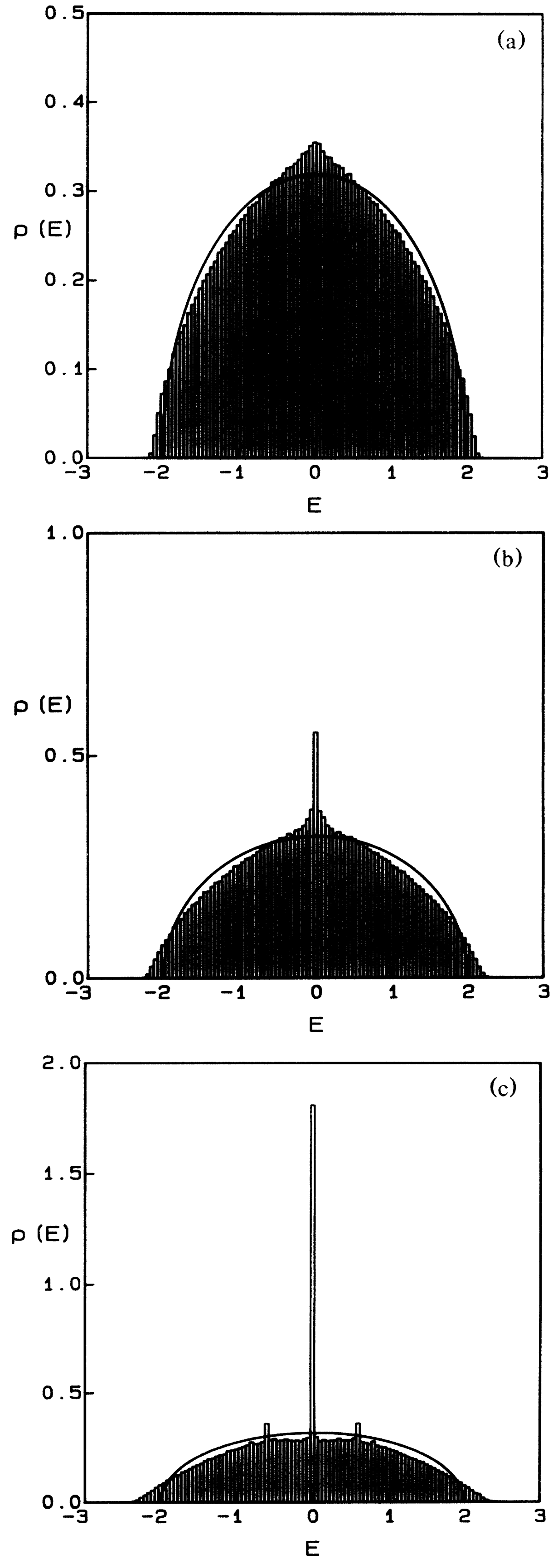

FIG. 1. Plot of the normalized averaged DOS $\langle\rho(E)\rangle$ together with the Wigner semicircle law for sparse random matrices of $N = 2000$ with three different values of p: (a) $p = 8$ from 200 matrices, (b) $p = 5$ from 100 matrices, and (c) $p = 3$ for 100 matrices.

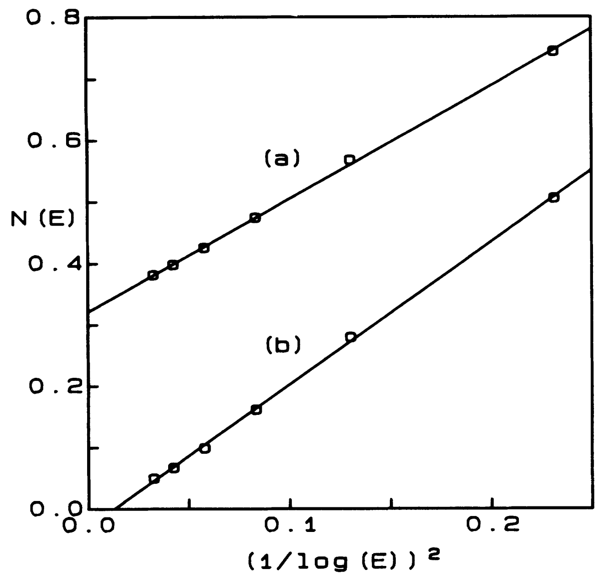

FIG. 2. Plot of the averaged integrated density of states $N(E)$ for E close to $E=0$ for, curve a, $p=5$ and, curve b, $p=3$. The straight lines imply that the singularity is of the form of Eq. (3).

when they are delocalized. In fact, for critical states $\langle[\delta n(E)]^2\rangle$ is still proportional to $\langle n(E)\rangle$ but with a proportionality index of about $\frac{1}{2}$ [17], a bound used as a localization criterion.

In order to estimate p_q we measured the sample-to-sample relative DOS fluctuations for various fixed small-p values. Our results will be presented in full elsewhere but here we report that they indicate a quantum percolation threshold of $p_q \simeq 1.4$, found as the concentration where the relative fluctuations were about $\frac{1}{2}$. From the correspondence of the SRME to the infinitely coordinated diluted Bethe pseudolattice the transitions of classical percolation at $p_c=1$ and quantum percolation at $p_q \simeq 1.4p_c$ are expected [18], in agreement with our result.

We have also studied the distribution function $P(S)$ of the nearest-energy-level spacings $S_n = E_{n+1} - E_n$, using a quite rigorous analogy which relates Wigner-Dyson statistics with delocalized states and ordinary Poisson

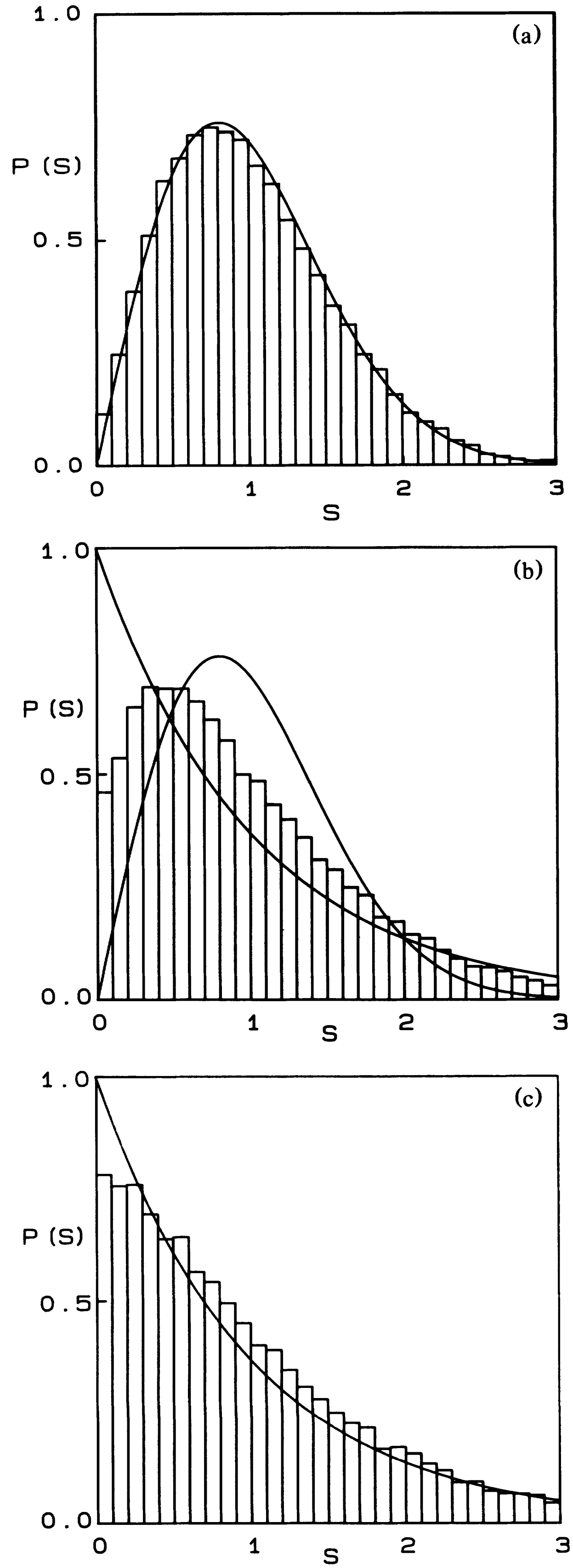

FIG. 3. (a) The calculated level-spacing distribution function for $N=1000$ and $p=3>p_q$. The data are in histogram form for 200 random matrices and cover the full energy range. The horizontal axis is in units of the local mean-level spacing and the solid curve is the Wigner surmise $P(S)=(\pi S/2)\times\exp(-\pi S^2/4)$, which implies the occurrence of a smooth, correlated spectrum exhibiting level repulsion associated with delocalized states. (b) As in (a) but for $p=1.4\simeq p_q$. The continuous lines are the Wigner surmise and the Poisson laws, respectively. (c) As in (a) but for $p=1<p_q$. The continuous line is the usual Poisson law $P(S)=\exp(-S)$ and the spectra are uncorrelated, obeying normal statistics corresponding to localized states.

statistics with localization, previously exploited for responsibility of the mesoscopic-physics fluctuation phenomena [13]. The calculations are done for various p values by obtaining the eigenvalues for many random runs, their total number being approximately 50000, and subsequently by deconvoluting the spectrum [6,19], in order to retain a constant DOS. This is equivalent to studying the distribution of the differences

$$\langle \mathcal{N}(E_{n+1})\rangle - \langle \mathcal{N}(E_n)\rangle = (E_{n+1}-E_n)\frac{\partial}{\partial E}\langle \mathcal{N}(E)\rangle ,$$

where $\langle \mathcal{N}(E)\rangle$ is the averaged IDOS at energy E. In Figs. 3(a)-3(c) the results for $P(S)$ are displayed and are shown to agree well with the corresponding Wigner surmise and the Poisson law for p above and below p_q, respectively. For $p \simeq p_q$ instead we obtain an intermediate distribution. From these results we conclude that the Wigner-Dyson universality is perfectly valid even for large deviations from the GOE limit, if $p > p_q$.

We summarize the main results for the SRME obtained in this Letter: (i) The DOS perfectly satisfies the semicircle law down to moderate values of p, and a crossover to a DOS with a presence of a $1/E$ singularity peak for $|E|$ near zero is seen by significantly lowering p. (ii) The validity of the GOE universality is demonstrated by the occurrence of the Wigner surmise even when drastically departing from the GOE, in the very sparse random matrix limit, as long as $p > p_q$. For $p < p_q$ the $P(S)$ approaches a Poissonian distribution. At the critical point we obtain an intermediate distribution interpolating smoothly between the two. Therefore, our results for $P(S)$ signify the wider validity of the Wigner-Dyson theory and also enable us to distinguish between localized and delocalized states. Moreover, they might be useful in giving justification for the presence of the correlated spectra needed to explain relaxation data in glasses [20] and in connection with the possibility of constructing a mean-field theory of Anderson localization.

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