Scenarios for the Nonlinear Evolution of Alpha-Particle-Induced Alfvén Wave Instability

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Various nonlinear scenarios are given for the evolution of energetic particles that are slowing down in a plasma and simultaneously excite the background plasma waves. Depending on the relationships between the source, the background damping, and the classical transport rate, either a steady state or pulsations arise. At the predicted saturation levels, anomalous particle transport is rather low. However, if the particle orbits are stochastic at the amplitude level needed to balance the growth rate with the wave trapping frequency, a phase space "explosion" occurs, giving enhanced transport.

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The problem of alpha-particle confinement under ignition conditions has been of considerable interest recently as there is concern that they can be anomalously lost due to their excitation of Alfvén waves [1-7]. Recent experiments with neutral beams [8,9] have established such behavior. The nonlinear consequences of this instability have been the topic of several theoretical treatments [5-7]. In this Letter we generalize the previous works by Berk and Breizman [5] (BB) to obtain a broader description of the nonlinear behavior of high-energy particles (which we will refer to as alpha particles; in deuterium-tritium fusion conditions this is a proper designation, though more generally these particles need only be superthermal and they can arise from beam injection, ion cyclotron heating, etc.).

In BB the nonlinear problem was considered as a generic problem where similar mathematics applies to the bump-on-tail electrostatic plasma instability or the universal instability drive that excites electrostatic drift waves or electromagnetic Alfvén waves. As the wave-particle interaction for the electrostatic plasma oscillation is a paradigm in nonlinear dynamics, we will discuss this problem in parallel with the mathematically "isomorphic" problem of alpha particles exciting Alfvén waves in a tokamak. What is required in these problems is to have a weakly damped wave existing in the background plasma in the absence of energetic particles. The energetic particles are injected at high energy, slow down by drag, and their pitch angles diffuse in velocity space through classical scattering. These classical processes establish an equilibrium with the source of energetic particles.

Instability will be possible if the shape of the alpha-particle distribution F_α is destabilizing in the vicinity of a phase-space region where particles resonate with the background wave. For the bump-on-tail instability, we require, in the vicinity of $\mathbf{k}\cdot\mathbf{v}=\omega$,

$$\frac{\mathbf{k}\cdot\mathbf{v}}{\omega}\frac{\partial F_\alpha}{\partial v^2}(\mathbf{v})>0\,, \tag{1}$$

with $\mathbf{k}$ the wave number, ω the wave frequency, and $\mathbf{v}$ the energetic particle velocity. For the universal instability in a tokamak we require, in the vicinity of $p\omega_\theta=\omega-n\omega_\varphi\equiv\bar{\omega}$,

$$\frac{p\,\partial F_\alpha/\partial r^2}{\omega\Omega_\alpha\partial F_\alpha/\partial v^2}\equiv\frac{\omega_{*\alpha}}{\omega}>1\,, \tag{2}$$

where p is an integer, Ω_α is the alpha-particle gyrofrequency, ω_θ is the poloidal transit frequency, ω_φ is the toroidal transit frequency, and it is assumed that $\partial F_\alpha/\partial v^2<0$. Because of toroidal symmetry, the wave amplitude is taken to be proportional to $\exp(in\varphi)$, with n an integer.

In BB a steady-state nonlinear wave was predicted when the classical transport of alpha particles is accounted for. The solution allows for a balance between the nonlinear alpha particle instability drive and plasma dissipation. In this Letter we show that such a solution requires the background damping to be sufficiently weak. However, for stronger background damping rates, we now show that the nonlinear solution is unstable. In this case a new nonlinear scenario emerges. The system no longer maintains a steady-state solution. Instead the response is that of pulsations, as described below.

Suppose $\gamma_L\gg\gamma_d,\nu_{\rm eff}$, where γ_L is the linear growth rate that would be predicted from the distribution function that forms from a classical relaxation process in the absence of excitations, γ_d is the dissipation rate of the excited wave caused by the background plasma, and $\nu_{\rm eff}$ is the rate of reconstruction of the unperturbed distribution function after it has been flattened in phase space by a nonlinear wave. Typically, pitch angle diffusion dominates this process, and in this case $\nu_{\rm eff}\approx\nu(\omega/\omega_b)^2$, where ν is the 90° velocity pitch angle scattering rate, and ω_b is the trapping frequency of resonant particles trapped in the wave.

Let us first suppose that $\nu_{\rm eff}\gg\gamma_d$. In this case the BB solutions are appropriate. In steady state a wave is found, where the power P_α, which is transferred from the alpha particles, is given by

$$P_\alpha\approx\gamma_L(\nu_{\rm eff}/\omega_b)E_{\rm wave}\,, \tag{3}$$

where $E_{\rm wave}$ is the energy of the wave. (For electrostatic plasma waves,

$$E_{\rm wave}=\int d^3r\,\overline{|\delta\mathbf{E}|^2}/4\pi\,,$$

where the bar refers to time average, $\delta\mathbf{E}$ is the perturbed electric field, and equal energy contributions are taken into account for perturbed electric field energy and perturbed kinetic energy. For Alfvén waves,

$$E_{\rm wave}=\int d^3r\,\overline{|\delta\mathbf{B}|^2}/4\pi\,,$$

where $\delta\mathbf{B}$ is the perturbed magnetic field and the equal contribution of perturbed kinetic energy is accounted for.) Generically, $\omega_b \propto \Phi^{1/2}$ with Φ a measure of the perturbed field amplitude [e.g., $\Phi=\delta E$ for plasma waves with $\omega_b^2=(e/M_a)k\delta E$]. This power is absorbed by background dissipation: $P_d=-2\gamma_d E_{\rm wave}$. Hence, with $P_a+P_d=0$, the saturated wave amplitude satisfies

$$\omega_b \approx \gamma_L \nu_{\rm eff}/\gamma_d \approx (\gamma_L \nu\omega^2/\gamma_d)^{1/3}\,. \tag{4}$$

As we assumed $\gamma_d<\nu_{\rm eff}$, we see that the relaxation process pumps the wave to an amplitude Φ that gives a trapping frequency higher than the linear growth rate. We further find $\nu_{\rm eff}/\gamma_d \approx (\nu_{\rm eff0}/\gamma_d)^{1/3}$, with $\nu_{\rm eff0}=\nu\omega^2/\gamma_L^2$. The significance of $\nu_{\rm eff0}$ will be clarified below. We also note that in this regime $\nu_{\rm eff0}>\nu_{\rm eff}>\gamma_d$.

If $\nu_{\rm eff}\ll\gamma_d$, the predicted trapping in Eq. (4) is lower than γ_L. In this case the nonlinear steady-state distribution function found in BB is unstable, basically to the same linear instability that exists in the unperturbed state. This observation readily follows from closely examining the response of linear theory. The linear growth rate for a smooth distribution function formed in the absence of nonlinear waves is proportional to a quantity D given by the following expression:

$$D=-\mathrm{Im}\int d^3v\,\frac{\mathbf{k}\cdot\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{\partial F_a}{\partial v^2}$$
$$=\pi\int d^3v\,\mathbf{k}\cdot\mathbf{v}\frac{\partial F_a}{\partial v^2}\delta(\omega-\mathbf{k}\cdot\mathbf{v})\,. \tag{5}$$

[For the Alvén wave problem there is a similar structure for D with

$$\mathbf{k}\cdot\mathbf{v}\frac{\partial F_a}{\partial v^2}\rightarrow(\omega-\omega_{*a})\frac{\partial F_a}{\partial v^2}G_{p,n}(\mathbf{v}),\quad \omega-\mathbf{k}\cdot\mathbf{v}\rightarrow\bar{\omega}-p\omega_\theta\,,$$

where $G_{p,n}(\mathbf{v})$ is a positive slowly varying function of phase space.] One readily demonstrates that $\gamma_L\propto D$.

Now in the case $\nu_{\rm eff}\ll\gamma_d$, the nonlinear distribution function found in BB is essentially the same as the unperturbed case, except in a small resonance region where particles are trapped in the wave. There the distribution is flattened over a phase-space region

$$\delta v\approx\omega_b/k\equiv\delta v_b \tag{6}$$

(note that it is shown in BB that for the Alvén wave problem δv transforms to a positionlike variable in the case $\omega\ll\omega_{*a}$, viz., $\delta v/v\rightarrow\delta r/r$). Outside this region virtually the same self-consistent F_a is obtained as in the unperturbed case. Hence, if one attempts to evaluate $D(\omega)$ in Eq. (5), with this locally flattened distribution function, one finds that though $D(\omega_0)\rightarrow 0$ with ω_0 the real frequency of the background oscillation, the value for $D(\omega_0+i\gamma_L)$ is hardly changed at all from the smooth case [the difference is $\mathcal{O}(\omega_b/\gamma_L)$]. Hence the BB steady-state solution is unstable for sufficiently large γ_d, viz., $\gamma_d\gg\nu_{\rm eff}>\nu_{\rm eff0}$.

This result indicates that the nonlinear response in the $\gamma_d\gg\nu_{\rm eff0}$ limit cannot be a steady state. Instead the following pulsation scenario seems consistent. Suppose the linear instability with the smooth F_a distribution develops at the rate γ_L. The distribution function for the bump-on-tail instability would initially look like the smooth solid line in Fig. 1, just when instability begins. Then, as basic and straightforward arguments indicate, the wave amplitude will grow until the trapping frequency of the wave reaches the linear growth rate γ_L (we define ω_{b0} as that trapping frequency in which $\omega_b=\gamma_L$). The wave flattens the distribution function in the resonant region which destroys the resonant particle drive, much in the same manner as described by O'Neil [10] and Mazitov [11] and it is depicted by the dashed curve in Fig. 1. However, with the background dissipation present, this wave will now damp according to the equation $dE_{\rm wave}/dt=-2\gamma_d E_{\rm wave}$. Simultaneously, the classical transport mechanism attempts to reconstitute the unstable distribution function at a rate $\nu_{\rm eff0}$ as the flattening of the distribution function only occurred in a phase-space region $\delta v/v_0\approx\omega_{b0}/\omega\approx\gamma_L/\omega$, where $\mathbf{k}\cdot\mathbf{v}_0=\omega$ [or $p\omega_\theta(\mathbf{v}_0)=\bar{\omega}$]. Thus the time for the wave energy to disappear is $1/\gamma_d$, while the time for the reconstitution is $1/\nu_{\rm eff0}$. After a time $1/\nu_{\rm eff0}$ the distribution is again ready to excite waves and grow to an amplitude where $\omega_b\sim\gamma_L$. During intermediate times, $1/\gamma_d<t<1/\nu_{\rm eff0}$, precursor instability may arise, for example, when the distribution is shaped

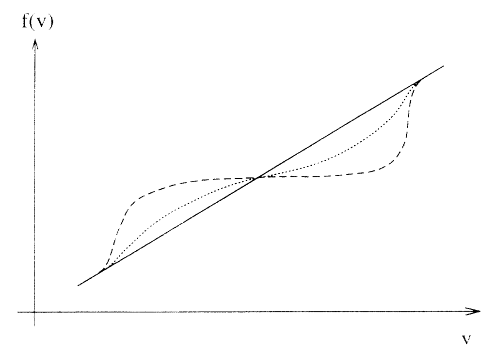

FIG. 1. Time behavior of the bump-on-tail distribution function near the resonant mode phase velocity. The solid curve indicates distribution just before its major relaxation, the dashed curve is just after the major relaxation, and the dotted curve is at an intermediate time during which the distribution is being reconstituted.

like the dotted curve in Fig. 1. Saturation by particle trapping with a trapping frequency $\omega_{b1} \approx \gamma_L \nu_{\text{eff}0} t$ will then occur. However, these precursor waves do not destroy the free energy of the distribution in the range

$$\omega_{b1}/\omega_{b0} < |\mathbf{k}\cdot\mathbf{v}-\omega|/\omega_{b0} < 1 .$$

Thus, low level precursor waves are expected prior to the largest "crash." After the largest crash, when $\omega_b \approx \omega_{b0} \approx \gamma_L$, the distribution is again flattened over the interval $\delta v \approx |\delta v_b|$, with δv_b defined in Eq. (6), and then the process described repeats itself with an overall period $\nu_{\text{eff}0}^{-1} \approx \nu^{-1}(\gamma_L/\omega)^2$.

The need for a pulsation scenario can also be explained in terms of energy balance. Over a long time scale, the average background dissipation can be estimated as $\gamma_d \bar{E}_{\text{wave}}$, with $\bar{E}_{\text{wave}}$ the time-averaged wave energy. This dissipation must be balanced by the free energy that is brought to the resonant region by collisions. In a time $1/\nu_{\text{eff}0}$ the free energy of the particles is built up and then converted to the maximum wave energy $E_{\text{wave max}}$ determined from the condition $\omega_b \approx \gamma_L$. Hence the estimate for the feed power into the wave is $\nu_{\text{eff}0} E_{\text{wave max}}$. Equating the feed power to the average dissipative power gives $\bar{E}_{\text{wave}} = (\nu_{\text{eff}0}/\gamma_d) E_{\text{wave max}}$, or equivalently

$$\bar{\omega}_b = \gamma_L (\nu_{\text{eff}0}/\gamma_d)^{1/4} , \tag{7}$$

where $\bar{\omega}_b$ is a trapping frequency based on the time-averaged wave energy. Since $\nu_{\text{eff}0}$ is assumed to be much less than γ_d, the average wave energy is much less than the maximum one. Such a condition can only be achieved with relaxation oscillations, as depicted by the solid curves in Fig. 2. Also note that $\bar{\omega}_b$ in Eq. (7) is larger than the saturation level predicted in Eq. (4), as $\nu_{\text{eff}0} < \gamma_d$. Further, as previously discussed [5], for $\nu_{\text{eff}0}/\gamma_d > 1$, the wave energy saturates at a level $E^*_{\text{wave}} = (\nu_{\text{eff}0}/\gamma_d)^{4/3} E_{\text{wave max}}$, as depicted by the dashed line in Fig. 2.

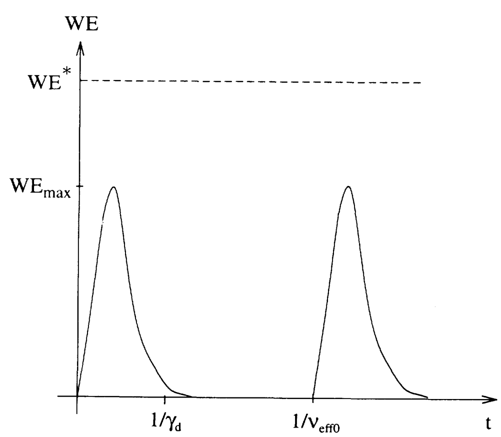

FIG. 2. Relaxation oscillations. If $\nu_{\text{eff}0} < \gamma_d$, relaxation oscillations arise as shown by solid curves. If $\nu_{\text{eff}0} > \gamma_d$, the wave energy (WE) saturates in steady state at a level $E^*_{\text{wave}} = (\nu_{\text{eff}0}/\gamma_d)^{4/3} E_{\text{wave max}}$.

This suggested scenario is valid in the tokamak problem if at the perturbation amplitude ω_b the alpha-particle orbits are not stochastic. Then it is easy to show that the radial spreading of a typical alpha particle, caused by its interaction with the pulsating field, is small. Hence the desired mechanism of heating the background plasma by collisions with the fusion produced alpha particles is attained.

On the other hand, if at the fluctuation level corresponding to $\omega_b \approx \omega_{b0}$ the stochasticity level is exceeded, catastrophic development is expected. This is because now orbits really diffuse and there are no longer barriers to maintain an overall "inverted" phase-space gradient in the vicinity of the resonance region. This is illustrated by allowing for multiple modes in the bump-on-tail instability. Below the critical amplitudes for mode overlapping, the situation is depicted in Fig. 3(a), where the distribution flattens in the shaded region, with an energy release proportional to the number of modes. The picture changes drastically, as in Fig. 3(b), when the resonances overlap. Then all the free energy at the inverted gradient is available to pump the wave to yet higher levels, thereby even increasing the particle diffusion. For Alfvén waves in a tokamak, resonance overlap may occur even for a single mode structure, because of the multiple resonances in the particle Hamiltonian. Further, the mode may be spatially spread, as typically occurs in the toroidal Alfvén

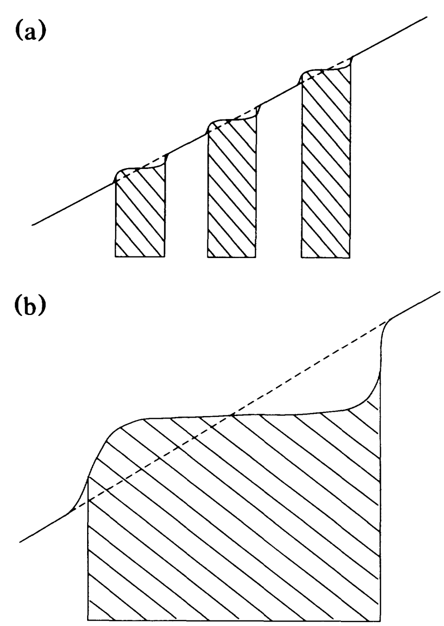

FIG. 3. Effect of resonance overlapping. In (a) modes do not overlap, and the relaxed distribution just has local flattening, with the general shape of the inverted equilibrium distribution preserved. When there is mode overlapping as in (b), the distribution flattens completely over the entire spectrum, with a much larger conversion of free energy to wave energy.

eigenmode where the mode is excited at different poloidal mode numbers throughout the radial profile [3]. Hence the alpha particles can then be either lost to the boundaries or, if the system is large enough, the distribution function is flattened to a profile that is stable to linear analysis.

To obtain a feel for the stochastic threshold we note that from Ref. [12] one finds that $\gamma_L/\omega \sim 5q^2\beta_\alpha$ for moderately high n modes, where β_α is the beta value of the alpha particles. For these modes the trapping frequency and stochasticity threshold have been reported in Ref. [13] to be

$$\frac{\omega_b}{\omega} \approx \left(\frac{\delta B_\theta}{B}\right)^{1/2}, \quad \frac{\delta B_\theta}{B} \sim \frac{1}{16nq},$$

respectively. Thus a very rough criterion for the onset of stochasticity is

$$\beta_\alpha > 1/20q^{5/2}n^{1/2}.$$

Though only rough scaling arguments are given here, these suggested scenarios seem compatible with experimental observation [8,9]. More careful quantitative studies are of course needed.

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