Effect of Emittance and Energy Spread on a Free-Electron Laser in the Gain-Focusing Regime

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A free-electron laser operating in the gain-focusing regime is discussed. The variation of growth rate, radius of curvature of wave fronts, filling factor, and efficiency with emittance and energy spread is derived. The results, which are based on the Vlasov-Maxwell system of equations, are obtained by minimizing a variational functional. When plotted as a function of emittance, the efficiency at maximum growth rate peaks at a nonzero value of emittance. For small values of energy spread, the efficiency at maximum growth rate increases with energy spread, in contrast to intuitive expectations.

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We present the results of an analytical study of the effect of emittance and energy spread [1–15] on a free-electron laser (FEL) in the gain-focusing regime [16–19] of operation. Based on the Vlasov-Maxwell equations a differential eigenvalue equation for the wave number k of the radiation is derived and solved by a variational technique. We make use of the scaled parameters of Refs. [11,15]. However, in contrast to Refs. [11,15], our trial function depends on a variational parameter. This parameter furnishes additional information about the radiation field; namely, the spot size and the radius of curvature of radiation wave fronts [18,19]. The stationarity condition imposed on the variational functional implies that our results are insensitive to the choice of the trial function [20]. The variation of growth rate, radius of curvature of wave fronts, filling factor, and efficiency with emittance and energy spread is displayed graphically.

The model consists of a matched electron beam and a matched radiation beam propagating along the z axis through a planar wiggler. The wiggler vector potential is given by $\mathbf{A}_w = A_w \cosh(k_w y)\sin(k_w z)\mathbf{e}_x$, where A_w is the amplitude, $2\pi/k_w$ is the period, and $\mathbf{e}_x$ is the unit vector along the x axis. The vector potential of the optical beam is given by $\mathbf{A}_s = \frac{1}{2}A_s(y)\exp[i(kz-\omega t)]\mathbf{e}_x + \text{c.c.}$, where ω is the angular frequency and A_s is the amplitude.

The equations of motion of an electron are derived from the Hamiltonian function $-p_z(y,p_y;t,-E;z)$ [21]:

$$p_z = \frac{E}{c} - \frac{m^2c^3}{2E}\left\{1 + \frac{p_y^2+P_x^2}{m^2c^2} + \frac{a_w^2}{2}[1+(k_w y)^2] + \frac{a_w a_s}{2i} f_B \exp[i(k+k_w)z - i\omega t] + \text{c.c.}\right\},$$

where $E \equiv \gamma mc^2$ is the energy of an electron of rest mass m and charge $-|e|$, t is the time, (P_x,p_y) are the momenta conjugate to the coordinates (x,y), $a_{w,s} = |e|A_{w,s}/mc^2$, $f_B = J_0(\xi) - J_1(\xi)$ is the usual difference of Bessel functions, and $\xi = (a_w/2)^2/(1+a_w^2/2)$.

In the absence of the optical field electrons perform betatron oscillations in the y-p_y plane. The area in this plane is the action $I \equiv \int\int dy\, dp_y/2\pi = H/k_\beta$, where $H = cp_y^2/2E + Ek_\beta^2 y^2/2c$ is the Hamiltonian for the transverse motion and $k_\beta = a_w k_w/\sqrt{2}\gamma$ is the betatron wave number.

The electron distribution function evolves according to the Vlasov equation. For the equilibrium distribution we choose

$$F(E,P_x,I) = n_{b0}\frac{\exp[-(\gamma-\gamma_0)^2/\sigma_\gamma^2]}{\sqrt{\pi}\sigma_\gamma mc^2}\delta(P_x)\frac{\exp(-\sqrt{2}I/a_w k_w mc\sigma_b^2)}{\sqrt{\pi}a_w k_w mc\sigma_b}, \tag{1}$$

where $E_0 = \gamma_0 mc^2$ is the mean energy, $\sigma_\gamma mc^2$ is the energy spread, and $n_b(y) = n_{b0}\exp(-y^2/2\sigma_b^2)$ is the spatial density, with peak value n_{b0} and width σ_b.

In the Coulomb gauge the wave equation for the optical field takes the form

$$\frac{d^2a_s}{dy^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]a_s - \frac{if_B^2}{n_b}\left(\frac{\omega_b a_w}{2c}\right)^2 \int \frac{d\Gamma}{\gamma^2}\frac{\partial F}{\partial \gamma}\int_{-\infty}^{0} d(\omega z'/c)\, a_s[y(z')]\exp[i\phi(z')] = 0, \tag{2}$$

where $\omega_b = (4\pi n_b|e|^2/m)^{1/2}$ and $d\Gamma = dP_x\, dp_y\, dE$. From the Hamiltonian functions, $-p_z$ and H, it follows that $\phi(z') = (k+k_w-\omega/c\beta_z)z'$ and $y(z') = y\cos(k_\beta z') + (cp_y/k_\beta E)\sin(k_\beta z')$, where

$$\beta_z^{-1} = 1 + [1 + a_w^2/2 + (P_x/mc)^2 + \sqrt{2}a_w k_w I/mc]/2\gamma^2, \tag{3}$$

and y and p_y denote the location of an electron in the transverse phase space at $z'=0$.

Equation (2) may be written as $\mathcal{L}a_s(y)=0$, where $\mathcal{L}$ is a linear operator. The eigenvalue k may be obtained by using the variational principle $\delta J=0$, where $J\equiv\int_{-\infty}^{\infty}dy\,\tilde{a}_s(y)\mathcal{L}\tilde{a}_s(y)$, $\tilde{a}_s(y)=\exp(-k_s y^2/2\zeta_R)$ is the trial function, $k_s=2\gamma_0^2 k_w/(1+a_w^2/2)$, $\zeta_R\equiv k_s(2/\sigma_s^2+ik_s/R)^{-1}$ is the variational parameter, σ_s is the spot size of the optical beam, and R is the radius of curvature of the wave fronts.

Evaluating the functional J and equating the result to zero leads to

$$\mu-\frac{k_{\beta0}/k_w}{8k_{\beta0}\zeta_R}+\frac{k_{\beta0}/k_w}{(2k_{\beta0}\zeta_R)^{1/2}}\left(\frac{\gamma_0 D}{\sigma_\gamma}\right)^2\sum_{r=-\infty}^{\infty}\int_0^{\infty}dx\left(1+\frac{2\sigma_\gamma}{\gamma_0}\xi\right)[1+\xi Z^*(\xi^*)]I_r^2\left(\frac{k_s\epsilon}{2k_{\beta0}\zeta_R}x\right)\exp\left[-\left(1+\frac{k_s\epsilon}{k_{\beta0}\zeta_R}\right)x\right]=0\,, \tag{4}$$

where $x=cI/k_{\beta0}E_0\sigma_b^2$, $\mu k_w=\omega/c-k$, $\epsilon=k_{\beta0}\sigma_b^2$ is the unnormalized emittance,

$$D=\left[\left(\frac{2\pi}{k_{\beta0}k_s}\right)^{1/2}\frac{\tilde{I}_b}{I_A}\frac{a_w^2}{1+a_w^2/2}\right]^{1/2}f_B\,,$$

$\xi=\gamma_0[-\mu+(1-\omega/\omega_s)-2rk_{\beta0}/k_w-(k_{\beta0}/k_w)k_s\epsilon x]/2\sigma_\gamma$, $\omega_s=ck_s$, $I_A=1.7\times10^4\gamma_0\beta_{z0}$, A is the Alfvén current, $\tilde{I}_b$ is the current per unit length, I_r is the modified Bessel function of order r, $Z(\xi)$ is the plasma dispersion function [14], ω_{b0} is the plasma frequency evaluated at the peak density n_{b0}, and $k_{\beta0}$ is the betatron wave number evaluated at γ_0.

Equation (4) and an equation obtained by equating to zero the derivative of Eq. (4) with respect to the variational parameter determine μ and ζ_R. Taking $1+(2\sigma_\gamma/\gamma_0)\xi\approx1$ in Eq. (4), the scaled parameters are μ/D, $(1-\omega/\omega_s)/D$, $k_{\beta0}\zeta_R$, $k_s\epsilon$, $k_{\beta0}/k_wD$, and σ_γ/γ_0D [11,22].

Figure 1 shows the scaled spatial growth rate $\mathrm{Im}\mu/D$, the scaled radius of curvature $k_{\beta0}R$, the filling factor σ_b/σ_s and the scaled efficiency η/D as functions of $k_s\epsilon$. In this example the electron beam is initially monoenergetic, i.e., $\sigma_\gamma=0$, the emittance being due to pitch-angle scattering.

All numerical results presented here correspond to the particular value of "detuning" $1-\omega/\omega_s$ that gives the maximum growth rate. In Fig. 1, for example, the detuning varies from point to point as the emittance changes.

The nonlinear efficiency η may be determined from the energy lost by electrons when they are trapped in the ponderomotive buckets. It may be shown that $\eta=2\gamma_0^2\langle\beta_z-\beta_{\mathrm{ph}}\rangle/(1+a_w^2/2)$, where $\beta_{\mathrm{ph}}=\omega/(k_w+\mathrm{Re}k)$ is the phase velocity of the buckets and $\langle\,\rangle$ indicates an average over the distribution in Eq. (1) [1]. Making use of Eq. (3), we find $\eta=-\mathrm{Re}\mu+(1-\omega/\omega_s)-(k_{\beta0}/k_w)k_s\epsilon$. It is shown elsewhere that this expression is in good agreement with numerical simulations as the detuning is varied [23]. Figure 1(d) shows that as $k_s\epsilon$ increases the efficiency at first increases, reaches a maximum (for both $k_{\beta0}/k_wD=1$ and 0.1), and eventually tends towards zero. We find that as $k_s\epsilon$ increases from small values the difference between the velocities of the *fastest growing* ponderomotive wave and the beam increases and this accounts for the initial increase in efficiency in Fig. 1(d) [8]. Eventually, however, the slowing down of the beam with increasing emittance predominates and the curve for the efficiency turns over and tends towards zero.

Our expression for efficiency can be used if the ponderomotive wave "sees" the electrons as a cold beam. An estimate of when this is valid is provided by the relative magnitude of the axial thermal velocity, $\beta_{z,\mathrm{th}}\equiv\langle(\beta_z-\langle\beta_z\rangle)^2\rangle^{1/2}$, and $\langle\beta_z-\beta_{\mathrm{ph}}\rangle$. Making use of Eq. (3) one finds

$$\frac{\beta_{z,\mathrm{th}}}{\langle\beta_z-\beta_{\mathrm{ph}}\rangle}=\left[2\left(\frac{\sigma_\gamma}{\gamma_0 D}\right)^2+\left(\frac{k_{\beta0}}{k_wD}k_s\epsilon\right)^2\right]^{1/2}\left(\frac{\eta}{D}\right)^{-1}. \tag{5}$$

Equation (5) indicates that for the dashed portion of the curves in Fig. 1(d) the ponderomotive wave is resonant with thermal electrons and the efficiency is expected to be significantly modified by *kinetic effects in the nonlinear stage of the interaction.*

To study the effect of energy spread, Fig. 2 shows the results, for $k_s\epsilon=0.1$, as a function of the energy spread σ_γ/γ_0D. Surprisingly, Fig. 2(d) shows that the efficiency is a monotonically increasing function of σ_γ/γ_0D. Equation (5) indicates that the dashed portions of the curves in Fig. 2(d) are expected to be significantly modified by *kinetic effects in the nonlinear stage of the interaction.* In their region of validity, both curves in Fig. 2(d) indicate a significant increase in the maximum efficiency with increasing energy spread, in contrast to intuitive expectations.

In Ref. [11] a variational technique is employed to study the effect of beam quality in an FEL. However, there is no variational parameter in the analysis and consequently there is no information about the filling factor or the curvature of the wave fronts (and no estimate for the efficiency). A further point that must also be noted concerns the detuning. We have stressed that at each point in Figs. 1 and 2 the detuning is adjusted to yield the maximum growth rate. For example, for $k_s\epsilon=1$, $\sigma_\gamma=0$, and $k_{\beta0}/k_wD=1$ (Fig. 1), we find $(1-\omega/\omega_s)/D=0.27$. This is roughly an order of magnitude smaller than the detuning indicated in Ref. [11]. Additionally, we find that the detuning is a sensitive function of the energy spread on the beam (Fig. 2).

We have examined the effect of emittance and energy spread on the spatial growth rate, radius of curvature of radiation wave fronts, and filling factor. The analysis is based on the Vlasov-Maxwell system of equations and a variational solution of the eigenvalue equation, with a trial function that depends on a variational parameter. We

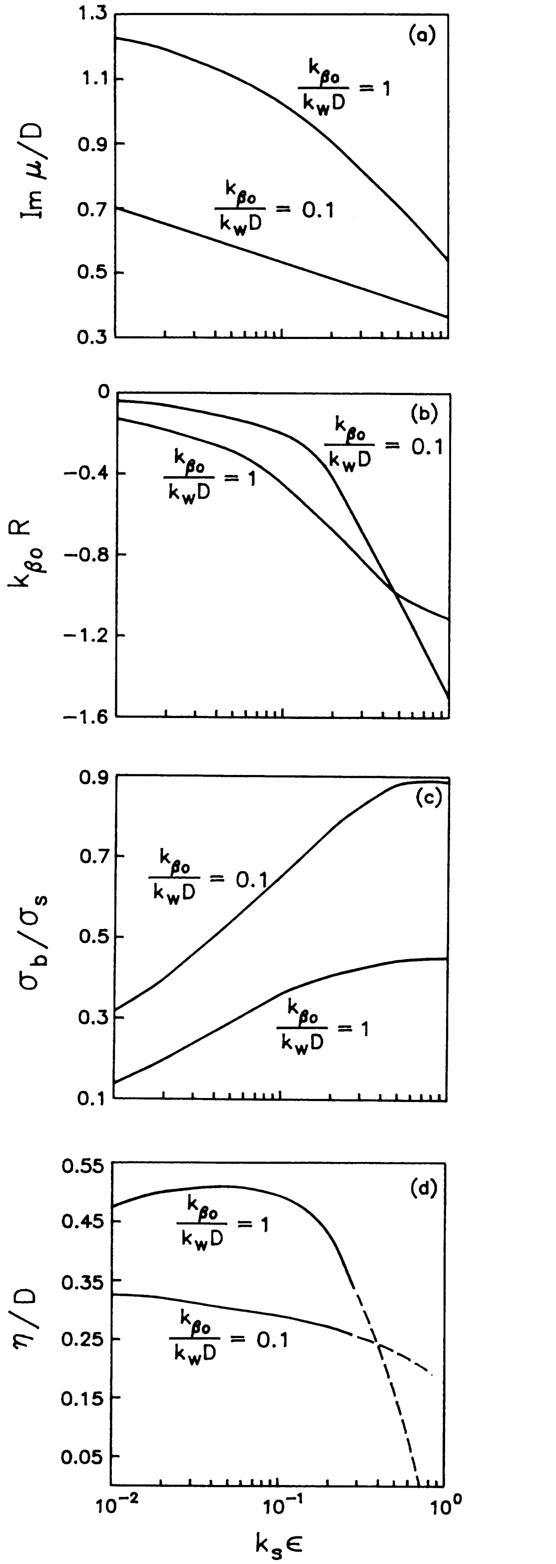

FIG. 1. Plot of $\mathrm{Im}\mu/D$, $k_{\beta 0}R$, σ_b/σ_s, and η/D vs $k_s\epsilon$ for a monoenergetic beam. The dashed portions of the curves in (d) lie in the regime where kinetic effects are expected to modify the efficiency.

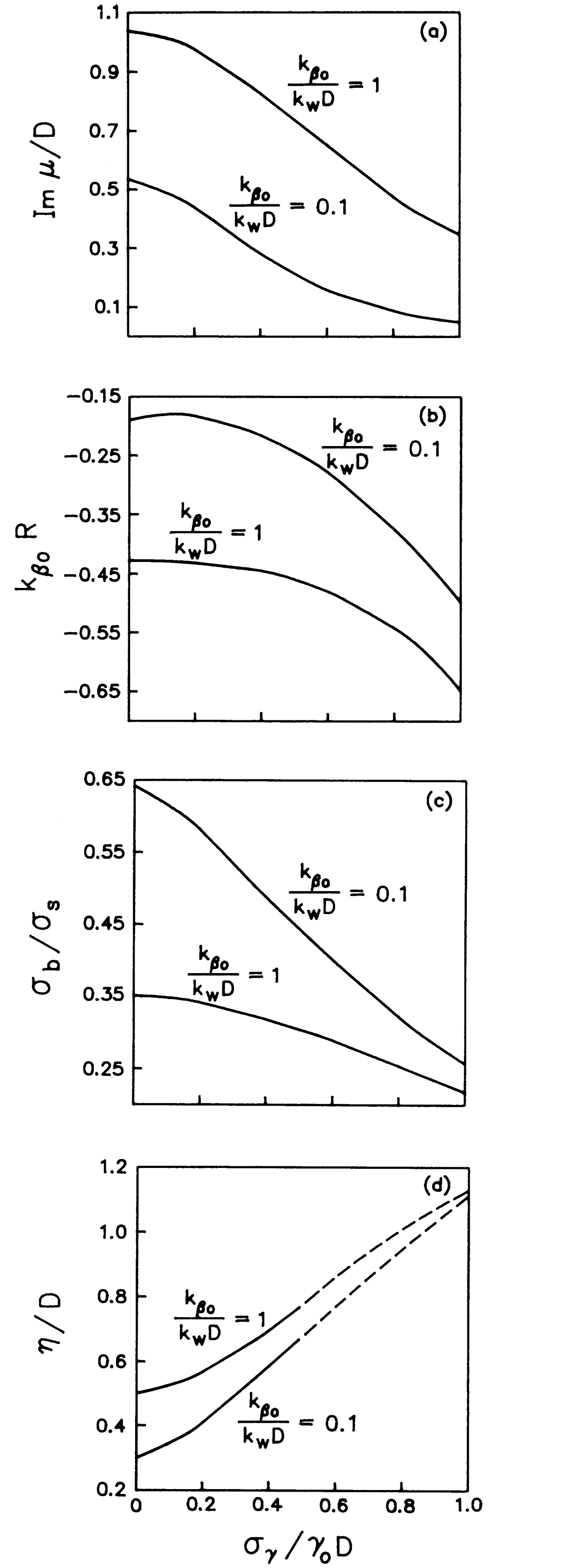

FIG. 2. Plot of $\mathrm{Im}\mu/D$, $k_{\beta 0}R$, σ_b/σ_s, and η/D vs $\sigma_\gamma/\gamma_0 D$ for $k_s\epsilon=0.1$. The dashed portions of the curves in (d) lie in the regime where kinetic effects are expected to modify the efficiency.

have found that (i) as a function of beam emittance, the efficiency peaks at a nonzero value of emittance and (ii) for small values of energy spread, the efficiency increases with energy spread on the beam.

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