Chaotic Scattering: A Toy Model for the Compound Nucleus

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(Received 5 February 1992)

A two-degrees-of-freedom Hamiltonian system is constructed and serves as a toy model for compound-nucleus scattering. The model contains both direct reactions and isolated and/or overlapping resonances. All its properties are in good agreement with the random matrix theory of compound-nucleus reactions. We attribute this agreement to the chaotic character of the process and its universality.

PACS numbers: 24.60.Lz, 24.60.Dr

Nuclear reactions at low energies, i.e., in the 10-MeV domain, can be separated into three major classes: direct, precompound, and compound-nucleus reactions. The striking dynamical differences between these processes are also reflected in the time scales typical for each reaction class. Direct reactions are the fastest processes; the associated reaction time τ_f is comparable to the time it takes the projectile to traverse the target nucleus. Compound-nucleus reactions are the slowest processes; the associated time scale $\tau_s = \hbar/\Gamma$ is given by the average width Γ of the compound-nucleus resonances, in both the domains of isolated and overlapping resonances. In this process, the system experiences many nucleon-nucleon collisions and thereby "equilibrates" before it decays. The internal equilibration time τ_{eq} is much shorter than τ_s. Precompound reactions are characterized by an intermediate time scale τ_i with $\tau_f \ll \tau_i \ll \tau_s$; here the nuclear system decays before it equilibrates.

The short time scale τ_f for the direct reaction signals that only a few degrees of freedom actively take part in the reaction. This is why simple *dynamical* nuclear models exist which successfully describe direct reactions: the optical model, the distorted-wave Born approximation, and the coupled-channels approach. In contradistinction, the compound-nucleus reactions involve many degrees of freedom which interact strongly. We are not aware of any realistic dynamical model for this type of reaction. Instead, a *statistical* model using random matrices of large dimensions has been successfully applied to calculate the mean value and other average properties of the compound-nucleus cross section, for all values of the ratio Γ/d (with d the mean spacing of compound-nucleus resonances of the same spin and parity) [1].

This discrepancy between *dynamical* modeling of direct reactions and *statistical* modeling of compound-nucleus reactions is not an accident. It reflects the near impossibility of calculating sufficiently accurately the details of the compound-nucleus excitation function in terms of the nucleon-nucleon interactions: Now and in the foreseeable future, we lack the detailed information on the nuclear forces and the technical tools needed to calculate the precise location and properties of compound-nucleus resonances (with a typical spacing of a few eV). The basic reason behind this impossibility is the extreme instability of the compound-nucleus resonances. Indeed, a change of the matrix elements of the nuclear Hamiltonian by a few eV—i.e., by an amount comparable to the mean level spacing in the neutron resonance region—is expected to qualitatively change the positions and partial widths of these resonances. The statistical modeling which is thus necessary employs the Gaussian orthogonal ensemble (GOE) of random matrices. This ensemble is known to reproduce correctly the spectral properties of dynamical systems which are classically chaotic [2]. The successful applications of the GOE to neutron resonances therefore suggest that nuclei can be related to chaos, and that compound-nucleus scattering is an example of quantum chaotic scattering.

This view is strongly supported by recent papers in which a comparison is made between the fluctuation properties of the scattering amplitudes calculated for chaotic dynamical systems, and those generated by the GOE [3,4]. The comparison has emphasized the rapid fluctuations in energy which happen on a scale of the mean level spacing and which correspond to the long-time behavior of the scattering process. The complete agreement found in this comparison, and the general arguments put forward for its justification [5], show that the GOE model for compound-nucleus scattering is, in fact, a universal model for quantum chaotic scattering, at least for the *rapid* fluctuations in energy of the elastic and inelastic amplitudes. This statement applies for all values of the ratio Γ/d.

In this Letter, we go beyond the statement just made. We wish to show that other concepts formulated in the framework of nuclear reaction theory likewise apply to chaotic scattering problems in general, and that the tools developed in the framework of nuclear physics are useful in analyzing a generic quantum chaotic scattering problem. More specifically, we show that an average, nonunitary S matrix can be defined in terms of a running average over an energy interval containing many resonances, that this average S matrix can be diagonalized by the unitary transformation introduced in Ref. [6], and that by transforming the entire (nonaveraged) S matrix in the same way, one obtains a new S matrix, which has all the properties of the GOE model, with transmission coefficients defined in terms of the diagonal elements of

the transformed average S matrix. We emphasize that these statements apply only to scattering problems which are chaotic in the classical limit (this is what we mean by "quantum chaotic scattering"). For scattering problems which are (close to) integrable and which display many resonances, these statements do not apply.

We perform this demonstration by constructing a model with two degrees of freedom which displays chaotic scattering and exhibits the main characteristics of low-energy nuclear reactions: elastic and inelastic scattering with closely spaced, isolated and/or overlapping resonances which have stochastic features. Needless to say, the parameters of our toy model have nothing to do with atomic nuclei. In this Letter, we only present the model and some results. A full description will be given elsewhere. With x_1,x_2 the two position coordinates and p_1,p_2 the conjugate momenta, our toy Hamiltonian is

$$H=\tfrac{1}{2}p_1^2-V_1\Theta(x_1)+V_2\Theta(x_1-l)+\tfrac{1}{2}p_2^2+\tfrac{1}{2}\omega^2x_2^2 + V_{\text{int}}\exp[a(x_1-x_2-l)]\,. \quad (1)$$

We restrict x_1 to the interval $x_1\le l$ by letting $V_2\to\infty$. For the other parameters, we use $V_1=20$, $\omega^2=0.6$, $V_{\text{int}}=0.1$, $a=3.5$, and $l=20\sqrt{2}$, and restrict x_2 to $-4\le x_2<4$. In the absence of any coupling between both degrees of freedom, i.e., for $V_{\text{int}}=0$, the "particle" with coordinate x_1 is scattered elastically by a "target" with internal coordinate x_2. The scattering is due to a square-well potential with depth V and length l. For all quantum calculations we took $\hbar=1$, so that the intrinsic motion of the target is modeled as a harmonic oscillator with energy spacing ω. For $V_{\text{int}}\neq0$, inelastic scattering is possible and ω is the energy difference between neighboring thresholds. The dimension of the S matrix is given by the number of open channels (and is thus energy dependent). "Compound-nucleus resonances" arise because in exciting the target, the particle may fall into a bound state of the square-well potential. This is the same mechanism as in actual compound-nucleus scattering. By a proper choice of parameters, it is possible to make the mean level spacing d of the resonances small compared to the threshold spacing ω. The interaction term in Eq. (1) was chosen in such a way that in the classical limit the system is chaotic. We have also studied cases where the systerm is (close to) integrable. Such cases are obtained by taking in the last term of Eq. (1) $V_{\text{int}}=10$ and $a=0.2$. For incident energies E between $(n-1)\omega$ and $n\omega$, with $\Lambda=n$ a positive integer, we have n open channels. For $n=1$, we deal with isolated resonances. The average width Γ of the resonances increases with n, and for $n\approx3$ we attain the situation $\Gamma\approx d$. Overlapping resonances were studied for $n=5$.

To show that the system (1) classically produces chaotic scattering, we have used the distribution $P(t)$ of the delay times t, with t defined as the difference between the time spent by the particle (coming in from the asymptotic domain $x_1\to-\infty$) in the interaction region (defined as $x_1>0$) and the free traversal time τ_f. For delay times $t\gg\tau_f$, chaotic systems have $P(t)\approx e^{-\gamma t}$ and integrable systems have $P(t)\approx t^{-\lambda}$ with γ,λ constants [7]. (For $t\approx\tau_s$, there is no universal behavior. The system-dependent "direct reactions" dominate.) Classical "reaction channels" are defined for $x_1\to-\infty$ in terms of the action-angle variables (I,θ) of the harmonic oscillator. We note that for $x_1\to-\infty$ and fixed total energy E, the action I is an integral of motion and defines a channel. To generate $P(t)$, we have calculated numerically a large set of trajectories, all starting at the same $\{x_1<0,E,I_i\}$ but at different starting angles θ_i. We have compared $P(t)$ with an exponential and have determined the energy-dependent constant $\gamma(E)$.

In the interval $0\le E\le6$, $P(t)$ was found to be an exponential; see Fig. 1. Moreover, the $\gamma_i(E)$ determined from different entrance channels i were, for the same E, compatible with each other, i.e., independent of the channel. We also calculated the angle θ_f at $x_1=0$ of the harmonic oscillator for trajectories leaving the interaction region (after scattering) for fixed I_i as a function of θ_i (inset, Fig. 1). The pattern is consistent with a multifractal set, as expected for hyperbolic chaotic systems [8].

Even for a system with two degrees of freedom like (1), the construction of the full S matrix for the quantum case is still a daunting task. To simplify it, we have adopted a standard procedure of nuclear physics. It will become

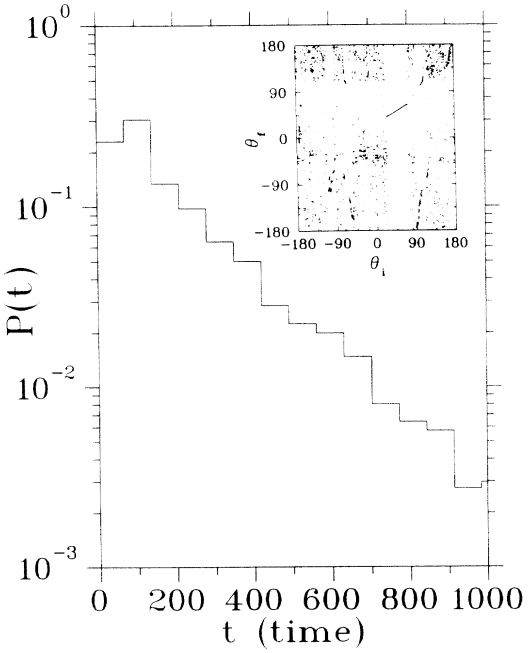


FIG. 1. Decay probability $P(t)$ as a function of time t in arbitrary units. Inset: Harmonic oscillator angle at the surface of section defined by $x_1=0$, for outgoing trajectories vs incoming ones; for details see text.

clear presently that this simplification does not alter the conclusions reached above concerning chaos for the classical case. We have decomposed the Hilbert space into a P space and a Q space, defined by the continuum spectrum (bounded spectrum, respectively) of Eq. (1) without coupling ($V_{\rm int}=0$), and have assumed that PHP is diagonal. This approximation suppresses direct reactions due to scattering of the particle via channel-channel coupling (mainly in the external region $x_1<0$), but still does allow for the direct reactions arising from processes going via the bounded space (in the interaction region $x_1>0$). Such direct contributions to S do indeed exist. With PHP diagonal, it is possible to write the scattering matrix $\tilde{S}_{ab}$ (with a,b the open channels) as [9]

$$\tilde{S}_{ab}(E)=e^{2i\delta_a}\delta_{ab}-2i\pi\langle\chi_a^{(-)}(E)|PHQ\frac{1}{E-QHQ-QHP(E-PHP)^{-1}PHQ}QHP|\chi_b^{(+)}(E)\rangle, \quad (2)$$

where $\chi_a^{(\pm)}(E)$ are the eigenfunctions of PHP, and δ_a the associated phase shifts. We have calculated $\tilde{S}_{ab}$ via Eq. (2) in terms of the matrix elements of QHQ, QHP, etc.

Universal behavior in classical chaotic scattering emerges for $t\gg\tau_s$. Quantum mechanically, this corresponds via the uncertainty principle to a rapid energy dependence of $\tilde{S}_{ab}(E)$. The nonuniversal behavior is found for $t\approx\tau_s$, i.e., in the smooth behavior of $\tilde{S}_{ab}(E)$. We have therefore divided $\tilde{S}_{ab}(E)$ into two parts. The smooth part $\langle\tilde{S}_{ab}\rangle$ is obtained by averaging $\tilde{S}_{ab}(E)$ over an energy interval ΔE containing many compound resonances. Typically, we have used $\Delta E\approx\omega/2$. The fluctuating part $\tilde{S}^{\rm fl}$ is defined as the difference $\tilde{S}_{ab}-\langle\tilde{S}_{ab}\rangle$. As is customary in nuclear physics, we identify $\langle\tilde{S}_{ab}\rangle$ with direct reactions and $\tilde{S}^{\rm fl}_{ab}$ with compound-nucleus scattering. We emphasize that the direct contribution is here defined by a running average and not, as is the case in stochastic models, by an ensemble average. We have "diagonalized" $\langle\tilde{S}\rangle$ by an unitary transformation, $(U\langle\tilde{S}\rangle U^T)_{ab}=\langle S_{aa}\rangle\delta_{ab}$, and have defined $S^{\rm fl}=U\tilde{S}^{\rm fl}U^T$. We define the transmission coefficients as usual by $T_a=1-|\langle S_{aa}\rangle|^2$. We have investigated whether the fluctuation measures for $S^{\rm fl}_{ab}$ depend on the transmission coefficients T_a in the same way as in the case of the random matrix model for the compound-nucleus scattering. In the latter case, this dependence is known.

The fluctuation measure of central interest is the autocorrelation function $C_{ab}(\epsilon)=\langle S^{\rm fl}_{ab}(E)S^{\rm fl*}_{ab}(E+\epsilon)\rangle$. For $\epsilon=0$, it is written in the form

$$\langle|S^{\rm fl}_{ab}(E)|^2\rangle=[(W_a-1)\delta_{ab}+1]\frac{T_aT_b}{\sum_c T_c}, \quad (3)$$

where the sum on c runs over all open channels. This is the Hauser-Feshbach formula [10], expressing Bohr's idea of the independence of formation and decay of the compound nucleus, with the elastic enhancement factor W_a.

We have tested Eq. (3) both for $a\neq b$ and for $a=b$. For $a\neq b$, we obtained good agreement between Eq. (3) and the cross section from the model (2). A much more sensitive test is obtained by comparing the elastic enhancement factor with the results obtained by modeling the long-lived resonances in terms of the GOE. For the function $C_{ab}(\epsilon)$, the GOE result is given in Ref. [1]. We have used this result and the right-hand side of Eq. (3) to define the elastic enhancement factors W_a for the GOE. A comparison of these with the elastic enhancement factors for our dynamical model (2) is given in Table I. The averaged cross section is taken over a finite energy interval so that the statistical error is expected to be of the order of $1/\sqrt{N}$, where n is the number of resonances in the considered interval. Since we have typical $N\approx 50$, the agreement between our toy model and the stochastic theory in Table I is very good. Another test is obtained by comparing the ϵ dependence of the square of the autocorrelation function, $|C(\epsilon)|^2$, as obtained in the framework of Eq. (2) with the GOE result as given in Ref. [1]. For the case of $\Lambda=5$ open channels, this comparison is made in Fig. 2. The agreement is seen to be very good for $|\epsilon|<$ twice the autocorrelation width where a finite range of data errors start to play a role. For the

TABLE I. Elastic enhancement factors W for $\Lambda=5$ open channels as a function of the transmission coefficients compared with $W^{\rm stoch}$.

Channel i	1	2	3	4	5
T_i	0.8	0.1	0.6	0.7	0.6
W	2.0	3.7	2.4	2.3	1.9
$W^{\rm stoch}$	2.20	2.97	2.11	2.14	2.11

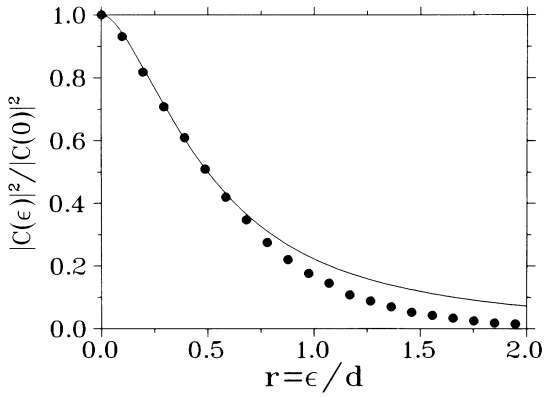


FIG. 2. Modulus squared of the normalized S-matrix autocorrelation function for $\Lambda=5$ open channels corresponding to $T_{\rm sum}\approx 2.8$ (circles), compared to the prediction of the stochastic approach (solid curve).

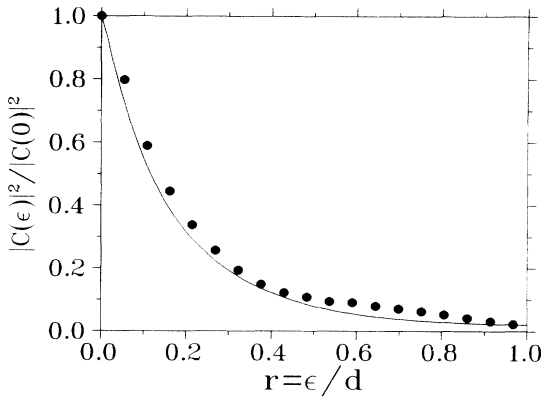


FIG. 3. Modulus squared of the normalized S-matrix autocorrelation function for one open channel, $T=0.45$ (circles), compared to the prediction of the stochastic approach (solid curve).

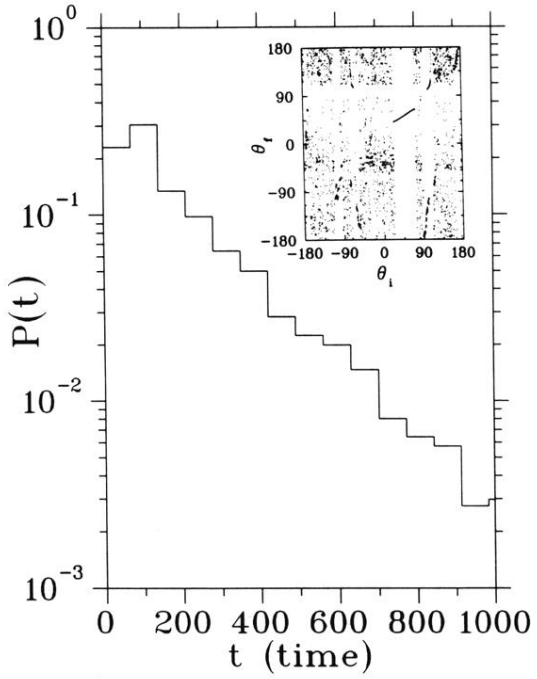
case of a single open channel, the comparison is shown in Fig. 3, again with very good agreement.

We have also compared the properties of an (almost) integrable system with the GOE predictions of Refs. [1,6]. We have found strong differences. For lack of space, we mention but one: In an (almost) integrable system, the elastic enhancement factor, defined by the right-hand side of Eq. (3), easily exceeds the upper bound of 3 of the GOE model and assumes values as large as 10.

In conclusion, we have shown that the toy model of Eq. (1), simplified as in Eq. (2), contains both direct and compound-nucleus reactions in the nuclear physics sense of these concepts. (Precompound reactions are absent. We attribute this to the small number of degrees of freedom.) Nondiagonal direct reaction contributions can be eliminated via the standard unitary transformation; the resulting direct elements S_{aa} define the usual transmission coefficients. Within the accuracy available (which is limited by a finite range of data errors), we have shown that averaged fluctuating cross section and correlation function $C(\epsilon)$ depend on these transmission coefficients exactly in the way predicted by the GOE model as worked out in Ref. [1]. This, in our opinion, demonstrates very clearly the universal applicability of concepts and ideas generated in the framework of nuclear physics.

The authors are indebted to T. H. Seligman and A. Müller for valuable discussions and suggestions. C.H.L. was supported by CNPq (Conselho Nacional de Desenvolvimento Científico a Tecnológico) (Brazil).

[1] J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, Phys. Rep. **129**, 367 (1985).
[2] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[3] R. Blümel and U. Smilansky, Phys. Rev. Lett. **60**, 477 (1988).
[4] C. H. Lewenkopf, A. Müller, and E. Doron, Phys. Rev. A **45**, 2635 (1992).
[5] C. H. Lewenkopf and H. A. Weidenmüller, Ann. Phys. (N.Y.) **212**, 53 (1991).
[6] H. Nishioka and H. A. Weidenmüller, Phys. Lett. **157B**, 101 (1985).
[7] P. Gaspard and S. A. Rice, J. Chem. Phys. **90**, 2225 (1989); **90**, 2242 (1989); **90**, 2255 (1989).
[8] U. Smilansky, in *Chaos and Quantum Physics,* edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier, New York, 1990).
[9] C. Mahaux and H. A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).
[10] W. Hauser and H. Feshbach, Phys. Rev. **87**, 366 (1952).



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