Spatial Coherence and Temporal Chaos in Macroscopic Systems with Asymmetrical Couplings

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Coupled map lattices with asymmetric short-range couplings are studied analytically and numerically. It is shown that with open boundary conditions these systems exhibit spatially uniform, but temporally chaotic states that are stable even in the thermodynamic limit. The stability of this state is associated with the appearance of a *gap* at zero wave number in the spectrum of the linear operator describing the fluctuations about the uniform state. The long-range order is unstable to noise. We calculate the finite *coherence length* of the chaotic state in the presence of weak noise.

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Can a long-range spatial order exist in a temporally chaotic macroscopic system with short-range interactions? Studies of this question using a variety of models, in particular coupled map lattices (CML), have shown that in the temporally chaotic states of these models the spatial correlations decay with a finite correlation length [1–5]. These results have been interpreted to show that generally temporal chaos in large systems leads to the loss of spatial coherence. The purpose of this Letter is to show that stable long-range coherence can coexist with temporal chaos even in macroscopic systems with short-range interactions (see also discussions in [6]).

We consider a one-dimensional CML [3,7,8] with nearest-neighbor couplings, given by the following equations:

$$x_j(n+1)=(1-\gamma_1-\gamma_2)f(x_j(n))+\gamma_1 f(x_{j-1}(n))+\gamma_2 f(x_{j+1}(n)), \qquad (1)$$

where γ_1 and γ_2 are the coupling constants, and $f(x)$ is a map of the interval onto itself. For all values of γ_1 and γ_2, Eqs. (1) have a spatially uniform solution of the form

$$x_i(n)=x(n), \quad x(n+1)=f(x(n)). \qquad (2)$$

The steady state of this solution is chaotic if the corresponding Lyapunov exponent ρ_0 defined by

$$\rho_0=\lim_{n\to\infty}\left(\prod_{l=1}^{n}|f'(x_0(l))|\right)^{1/n} \qquad (3)$$

is greater than 1.

Models of CML have been studied, mostly with symmetric forces, i.e., $\gamma_1=\gamma_2$, and periodic boundary conditions [1,2,4,5]. Under these conditions it is straightforward to show that the spatially uniform state is unstable in the chaotic regime, i.e., for $\rho_0>1$. In this paper we consider the asymmetric case $\gamma_1\neq\gamma_2$. We show that depending on the values of γ_1 and γ_2 the nature of the local instability may be of the *convective type* [9,10]. Hence it is very sensitive to the imposed boundary conditions. In particular, for *open boundary conditions* this coherent state will be stable in a nonmoving frame. Furthermore, we will show by numerical simulations that for these boundary conditions the uniform state is *globally* stable.

To study the stability of the uniform state we linearize Eqs. (1) about the uniform solution. To linear order in the deviations $\xi_j(n)=x_j(n)-x(n)$, Eqs. (1) are

$$\xi_j(n+1)=f'(x_0(n))[(1-\gamma_1-\gamma_2)\xi_j(n)+\gamma_1\xi_{j-1}(n)+\gamma_2\xi_{j+1}(n)]. \qquad (4)$$

First we consider periodic boundary conditions $\xi_j(n)=\xi_{j+N}(n)$. Then for large n the solution of Eqs. (4) can be written as

$$\xi_j(n)=\sum_{m=1}^{N}C_m\lambda^n(k_m)e^{ik_mj}, \qquad (5)$$

where $k_m=2\pi m/N$ and $m=0,\pm1,\pm2,\ldots$, and the eigenvalue spectrum $\lambda(k)$ is given by

$$\lambda(k)=\rho_0[1-\gamma_1(1-e^{-ik})-\gamma_2(1-e^{ik})]. \qquad (6)$$

Stability of a mode implies that $|\lambda(k)|<1$. We are interested in the limit of a large system, $N\to\infty$, with fixed γ_1 and γ_2. In this case, the spectrum λ_m given by Eq. (6) forms a continuous band, and in particular $\lambda(k)\to\rho_0$ for $k\to0$. Hence if $\rho_0>0$ not only is the uniform mode, $m=0$, unstable but there is a band of nonuniform unstable modes, implying that under small perturbation the spatial coherence of the system will be destroyed.

To study the nature of this instability we consider the evolution of a smooth initial perturbation, i.e., we solve Eqs. (5) assuming $C_m\to0$ for $m\to\infty$. Converting the sum over m to an integral over k and evaluating it in the long time limit using the saddle-point method, one finds that $\xi_j(n)\propto\lambda^n(k_0)$. The *complex* wave number k_0 is determined by the saddle-point condition $\partial\ln\lambda(k)/\partial k=0$ [11]. Using Eq. (6) yields $\exp(2ik_0)=\gamma_1/\gamma_2$, and

$$\lambda(k_0)=\rho_0[1-\gamma_1-\gamma_2+2(\gamma_1\gamma_2)^{1/2}]=\rho_0[1-(\gamma_1^{1/2}-\gamma_2^{1/2})^2]. \qquad (7)$$

For values of γ_1 and γ_2 such that $\lambda(k_0)<1$ the instability is convective: At each point j, $\xi_j(n)$ decays exponentially with n for large n, although the perturbation grows with time in an appropriate moving frame.

The prediction regarding the decay of the perturbation at any point ignored the boundary effects. In a system with periodic boundary conditions even convective instability will eventually lead to a destruction of the uniform state because a moving growing perturbation will always return to the origin. However, for open boundary conditions convective instability will not destroy the uniform state, as is shown below.

In the open boundary conditions, Eqs. (1) hold for all j except for $j=1$ and N where the terms proportional to γ_1 and γ_2, respectively, are absent:

$$x_1(n+1)=(1-\gamma_2)f(x_1(n))+\gamma_2 f(x_2(n)),$$

$$x_N(n+1)=(1-\gamma_1)f(x_N(n))+\gamma_1 f(x_{N-1}(n)).$$

The same holds for the linearized equations, Eqs. (4). The uniform solution, Eq. (2), still exists and is unstable to *uniform* fluctuations, the eigenvalue of which is again $\rho_0>1$. However, the remaining $N-1$ nonuniform eigenmodes of Eqs. (5) change. They are given by

$$\xi_j(n)=\rho^n(k_m)e^{\delta j}\cos(k_m j+\varphi_m), \qquad (8)$$

where $k_m=\pi m/N$, $m=1,2,\ldots,N-1$, $2\delta=\ln(\gamma_1/\gamma_2)$, and $\tan(\varphi_m)=[1-(\gamma_1/\gamma_2)^{1/2}\cos(k_m)]/\sin(k_m)$. The eigenvalue spectrum of these modes is

$$\rho(k)=\rho_0[1-\gamma_1-\gamma_2+2(\gamma_1\gamma_2)^{1/2}\cos k]. \qquad (9)$$

For $\gamma_1\neq\gamma_2$ this spectrum possesses a gap at $k=0$, since $\rho(k\to 0)\to\lambda(k_0)$, of Eq. (7), which is *less* than the $m=0$ eigenvalue ρ_0. In particular in the convective regime, i.e., for γ_1 and γ_2 such that $\lambda(k_0)<1$, all the $N-1$ nonuniform fluctuations are *stable*. The only instability which is left is the instability to uniform fluctuations which is inherent in the chaotic nature of the system. This instability, however, will not destroy the spatial coherence of the system.

The above predictions have been confirmed in numerical simulations of Eqs. (1) with open boundary conditions. We have used the logistic map $f(x)=\epsilon-x^2$ with $\epsilon=1.67$ which is in the chaotic regime $\rho_0=1.26$. Figure 1 shows the temporal evolution of the system starting from randomly nonuniform initial conditions. The coupling constants are $\gamma_1=0.7$ and $\gamma_2=0.1$ for which $\lambda(k_0)<1$. As is demonstrated in the figure, the left edge is a source of a synchronizing front that propagates with a finite velocity and leaves behind it a completely synchronized regime. The velocity of the front equals $\gamma_1-\gamma_2$ (see, e.g., [12]). Thus, the basin of attraction of the uniform state is big, and probably covers most of the space of initial conditions.

Figure 1 shows that the synchronizing front does not propagate throughout the system. In the sufficiently long lattice it stops before reaching the right edge, creating a domain of synchronized state with a *finite* coherence length, l_c. This phenomenon is the result of the numerical noise. Indeed, it is found numerically that l_c increases with the numerical precision, as demonstrated in Fig. 2.

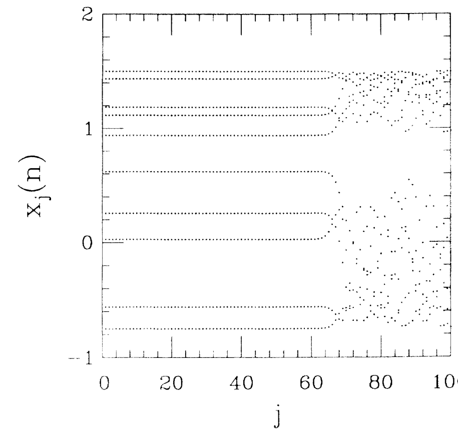

FIG. 1. Iterations of Eqs. (1) with $f(x)=1.67-x^2$, $\gamma_1=0.7$, $\gamma_2=0.1$, and $N=100$, starting from 10, randomly chosen initial conditions. We show the results after $n=1000$. The results reveal a coherent domain of length $l_c\approx 55$. The coherence length was defined from the condition $\max_{1<j<l_c}|x_{j+1}(n)-x_j(n)|\leq 10^{-6}$.

The sensitivity to the numerical noise indicates that the synchronized state is unstable to dynamic local noise. It is indeed expected that a weak noise will result in a finite coherence length. The reason is that each of the chaotic maps responds strongly to the perturbation of the noise. Maps near the left edge remain strongly synchronized with the map at the edge since the "synchronizing force" from the left maps is stronger than the noise. However, as we move along the chain the desynchronization caused by the noise increases and is completely destroyed at l_c. As will be calculated below, in the case of weak noise, $l_c\propto\ln\sigma$, where σ denotes the amplitude of the noise. On the other hand, the *width* of the domain wall separating the synchronized and the completely unsynchronized

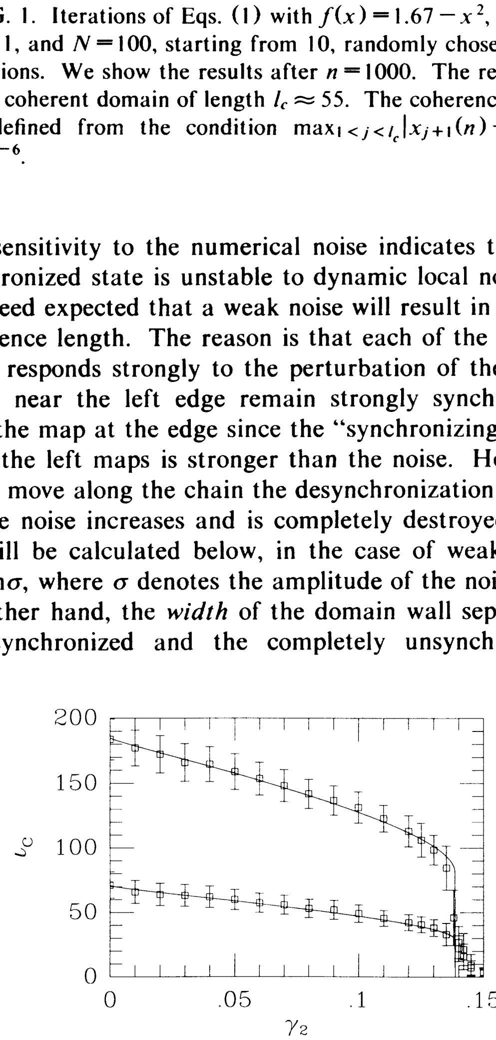

FIG. 2. The dependence of l_c on γ_2. All other parameters are the same as for Fig. 1. The data points represent estimates of l_c based on the results of numerical simulations of Eqs. (1), with γ_2 varying between 0.01 and 0.15. The upper part corresponds to simulations with a quadratic precision (32 digits after decimal point), and the lower to double precision (16 digits). Solid lines represent the best fit by Eq. (15) with $\sigma_0=1-\lambda(k_0)$ and σ a fitting parameter.

parts of the lattice is rather sharp. It is of order a few lattice constants.

To estimate the effect of a weak noise, we add to the right-hand side (RHS) of Eqs. (1) terms $\eta_j(n)$ representing a white uniformly distributed noise with a width σ. Assuming $\sigma \ll 1$ we evaluate the perturbation of the uniform state, Eq. (2), in the linear regime, using Eqs. (4) with the noise $\eta_j(n)$ added to its RHS. The coherence is defined as the minimal length l for which $\langle \xi_l^2(n\to\infty)\rangle \approx 1$. Solving Eqs. (4) with the added noise we find

$$\langle \xi_j^2(n\to\infty)\rangle = \sigma^2 \sum_{m_1=1}^{N}\sum_{m_2=1}^{N} \psi_{m_1}(j)\psi_{m_2}(j)\frac{Z(m_1,m_2)}{1-\rho_{m_1}\rho_{m_2}}, \quad (10)$$

where

$$Z(m_1,m_2)=\sum_{j=1}^{N}\psi^{\dagger}_{m_1}(j)\psi^{\dagger}_{m_2}(j). \quad (11)$$

Here $\psi_m(j)$ are the right eigenmodes of the linear operator defined by Eqs. (4), i.e., $\psi_j=\exp(\delta j)\cos(k_m j+\varphi_m)/\sqrt{2N}$ [see Eqs. (8)] and $\psi^{\dagger}_m(j)$ are their conjugate left eigenmodes. In the limit of small σ, j can be assumed to be large, and the sums of Eqs. (10) can be approximated as

$$\langle \xi_j^2(n\to\infty)\rangle \approx \sigma^2 e^{2\delta j}\,\mathrm{Re}\int_0^{k_c} dk\,k\frac{e^{i2kj}}{1-\rho^2(k)}, \quad (12)$$

where k_c is a cutoff of order 1. In the limit of large j this integral is dominated by the pole in the complex plane, i.e., $k=i\kappa$, $\kappa>0$, where $\rho(i\kappa)=1$ yielding

$$\langle \xi_j^2(n\to\infty)\rangle \propto \sigma^2 e^{2j(\delta+\kappa)}. \quad (13)$$

Using Eq. (9) and the definition of δ we obtain

$$\exp(\delta+\kappa)=\frac{1-\rho_0(1-\gamma_1-\gamma_2)-([1-\rho_0(1-\gamma_1-\gamma_2)]^2-4\gamma_1\gamma_2\rho_0^2)^{1/2}}{2\gamma_2^{1/2}\rho_0}. \quad (14)$$

Finally, equating Eq. (13) to 1 yields

$$l_c \approx \frac{\ln(\sigma/\sigma_0)}{\delta+\kappa}. \quad (15)$$

The constant σ_0 is roughly equal to $1/|\ln[1-\rho(k\to 0)]|$. Near the critical point it diverges, causing the vanishing of l_c. The dependence of l_c on $\ln\sigma$ is caused by the spatial amplification of the small perturbations due to the convective instability. The effect of noise on spatial coherence has also been discussed in [3,5].

The analytical result for l_c agrees well with the numerical simulations, as shown in Fig. 2. In the present case l_c drops to zero at $\gamma_2\approx 0.138$ where $\rho(k\to 0)=\lambda(k_0)\to 1$. Note, however, that the vanishing of l_c is relatively sharp in agreement with the predicted logarithmic singularity.

In conclusion, we have shown that a CML with sufficiently asymmetric short-range couplings and open boundary conditions can have a stable spatially uniform but temporally chaotic state. The signature of this state is a big gap, near $k=0$, in the spectrum of Lyapunov exponents of the linear operator characterizing the evolution of the fluctuations. These results extend those previously obtained for the purely unidirectional chain $\gamma_2=0$ [7]. This is a simple case in that a localized perturbation propagates with a finite velocity to the right and does not affect at all the maps behind it. In the present case where $\gamma_2\neq 0$ the perturbation also diffuses backwards. Nevertheless, we have shown here that if the perturbing front propagates sufficiently fast its tail in the backward direction is exponentially small and may not be able to destabilize the maps despite their exponential sensitivity to perturbations.

It is straightforward to generalize our results to CML in higher dimensions. An interesting question is whether, in general, either the asymmetry in the interactions or the open boundary conditions are necessary prerequisites for the emergence of long-range-ordered chaotic states. We believe that this phenomenon may exist also in systems with symmetric interactions, since the underlying dynamical state can spontaneously break the symmetry and give rise to convective instabilities. Furthermore, it is not inconceivable that in such cases the long-range spatial order may be stable even with periodic boundary conditions. This will happen if the propagating perturbation fronts and those reflected from the boundaries will annihilate each other, leaving intact the spatial order. A similar mechanism of stabilizing spiral states was recently shown for the complex Ginzburg-Landau model [13]. These questions are under current investigation.

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