Statistical Theory of Coulomb Blockade Oscillations: Quantum Chaos in Quantum Dots

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We develop a statistical theory of the amplitude of Coulomb blockade oscillations in semiconductor quantum dots based on the hypothesis that chaotic dynamics in the dot potential leads to behavior described by random-matrix theory. Breaking time-reversal symmetry is predicted to cause an experimentally observable change in the distribution of amplitudes. The theory is tested numerically and good agreement is found.

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Recently it has become possible to fabricate nearly isolated islands of two-dimensional electron gas on the surface of high-mobility semiconductor heterostructures, and probe their quasibound states by measuring tunneling conductance across the "quantum dot" [1-3]. A typical semiconductor dot is a few μm across, contains ~ 100 electrons, and is expected to have an irregular shape due to fluctuations in the electrostatic confinement potential. The tunneling spectroscopy of these novel objects is done by measuring conductance versus gate voltage (which varies the electron density and hence the Fermi energy ε_F). Varying ε_F also varies the equilibrium electron number, N, on the dot and so one probes successively not excited states of the dot for fixed N, but ground and low-lying states of the dot for each N. Thus the spacing $\Delta\varepsilon_F(N)$ between successive resonances is determined by *both* the additional Coulomb repulsion energy for adding an electron to the dot [taken to be simply $(2N-1)e^2/2C$, where C is the capacitance of the dot] *and* the splitting of the effective single-particle levels ε_N due to confinement. Within the conventional models [2,4] it follows that the resonance spacing is $\Delta\varepsilon_F=\varepsilon_N-\varepsilon_{N-1}+e^2/C$. If, as in the recent experiments [1,2], $e^2/C\gg\Delta\varepsilon=\varepsilon_N-\varepsilon_{N-1}$, approximately equally spaced peaks are observed (once $kT<e^2/C\sim 4$ K). Note that the tunneling is suppressed between resonances primarily by the charging energy, a phenomenon known as "Coulomb blockade" (CB).

Although many aspects of this resonance phenomenon are now understood theoretically [3-5], one of its most striking features, order-of-magnitude fluctuations in the amplitude of adjacent peaks [1,2] at low magnetic field, has been given no explanation. In this Letter we propose that these fluctuations arise from the chaotic nature of the eigenstates of irregular quantum dots, and calculate the statistical properties of the resonance amplitudes using random-matrix theory. We obtain a one-parameter distribution of peak heights which can be directly compared to histograms of the experimental resonance amplitudes as has been done previously for the analogous phenomenon of Porter-Thomas level-width fluctuations in neutron scattering [6,7]. Interesting new features arise in this system, however, due to thermal effects and the possibility of breaking the fundamental symmetries determining the statistics.

We focus on the regime in which $kT<\Delta\varepsilon$ (typically $\Delta\varepsilon\sim 0.5$ K), so that only one single-particle level contributes to each resonance [4,5]. Ignoring correlation effects with the Fermi sea [8] one has at $T=0$ the standard Breit-Wigner line shape with a total width $\Gamma_N=\Gamma_N^l+\Gamma_N^r$ (where $\Gamma_N^{l,r}$ are partial decay widths into left and right leads). However, when $kT\gg\Gamma_N$ (the typical case) the thermal rounding of the Fermi function $f(\varepsilon_F-E,T)$ leads to a resonance function [4] $g(\varepsilon_F-E_N,T)=-(e^2/h)A_N f'(\varepsilon_F-E_N,T)$, where $A_N=\Gamma_N^l\Gamma_N^r/\pi(\Gamma_N^l+\Gamma_N^r)$ is the *area* under the resonance. It follows that in this regime all resonances have the same width $\sim kT$ [1] but the amplitude depends on the $T=0$ decay widths,

$$g_{\max}=\frac{e^2}{h}\frac{\Gamma_N^l\Gamma_N^r}{4\pi kT(\Gamma_N^l+\Gamma_N^r)}\equiv\frac{e^2}{h}\frac{\bar\Gamma}{4\pi kT}\alpha_N\,, \tag{1}$$

where $\alpha_N=\pi A_N/\bar\Gamma$ is the amplitude normalized by the mean resonance width. Hence the large observed amplitude fluctuations imply that the decay widths fluctuate substantially on the scale $\Delta\varepsilon$.

Such large, nonmonotonic variation cannot be due to the barrier penetration factors which should be monotonic in energy and slowly varying on scale $\Delta\varepsilon$. Moreover, recent experiments [9] show that a magnetic field $B\sim 500$ G completely rearranges the amplitude pattern although such a field is too weak to affect substantially the tunneling rates. Thus the fluctuations must arise from spatial variations in the amplitude of the quasibound states inside the dot, and we will neglect randomness or complexity in the tunnel barriers below. The occurrence of fluctuations reminiscent of a random system is initially surprising for these high-mobility GaAs systems, which show ballistic behavior [2] at energies above the tunneling barriers. However, recent theoretical results [10] show that chaotic scattering in such devices leads to conductance fluctuations similar to those of highly disordered metals, and more generally the study of quantum chaos demonstrates that even modest complexity of shape, exemplified, e.g., by the stadium or Sinai billiards [11], will lead to effectively random behavior described by Wigner-Dyson statistics. We assume that such weak shape fluctuations exist in the dot and calculate the statistical distribution of amplitudes α_N, $\mathcal{P}(\alpha)$, from random-matrix theory [6,7].

First, Γ_N^l and Γ_N^r can be related to the bound states of the isolated dot via R-matrix theory [12] adapted to the quasi-1D case. We assume that the dot is coupled to infinite uniform leads through identical tunnel barriers. Outside the barriers we may expand the scattering wave solutions $\psi^+(x,y)$ at energy $E=\varepsilon_F$ in terms of products of longitudinal plane waves with wave vector k_n and transverse wave functions ϕ_n with subband energy ε_n (excluding exponentially growing terms when $\varepsilon_n > E$). We may also expand $\psi^+(x,y)=\sum_\lambda^\infty A_\lambda X_\lambda(x,y)$, where $\{X_\lambda\}$ is a complete set of eigenstates with energies ε_λ for the finite region of the dot plus tunnel barriers. Using current conservation a connection may be derived between $\{X_\lambda\}$ and $g(E)$, and in the "single-level approximation" [12] between a single X_λ and the resonance near ε_λ [13]. One finds that the partial width for decay into the nth open channel in the left lead (beginning at $x=0$) is

$$\Gamma_\lambda^l=|\gamma_{\lambda,n}^l|^2=\frac{\hbar^2 k}{2m}\left|\int_0^W dy\,\phi_n(y)X_\lambda(0,y)\right|^2, \qquad (2)$$

where $\hbar k=\sqrt{2mE}$. A general treatment for M open channels in each lead is straightforward, but since the leads will typically narrow smoothly near the tunneling barrier a single channel in each lead will have the shortest tunneling distance and will dominate the resonance.

We can define a reduced width $|\tilde{\gamma}_{\lambda,n}|^2=\Gamma_{\lambda,n}/P_{\lambda,n}$, where $P_{\lambda,n}\approx P(E-\varepsilon_n)$ is the barrier penetration factor which is a smooth function of energy. In this work we always take $P(E)$ the same on left and right (hence $\bar{\Gamma}^l=\bar{\Gamma}^r=\bar{\Gamma}/2$; the case $\bar{\Gamma}^l\neq\bar{\Gamma}^r$ will be treated elsewhere [13]). In this case α of Eq. (1) only depends on $\tilde{\gamma}_{\lambda,n}$ which is sensitive to the nature of the quasibound state, and has the same form as $\gamma_{\lambda,n}$ with the wave function *inside* the tunneling barrier $X_\lambda(a,y)$ replacing that outside [13]. Although Eq. (2) is derived for single-electron potential scattering, the approach can be generalized to many-electron wave functions [12] and will lead to the same *statistical* predictions for CB fluctuations even if residual interactions (beyond the charging energy) are important.

First we calculate the distribution $P(\Gamma)$ assuming that the Hamiltonian of the isolated dot is described at $B=0$ by the Gaussian orthogonal ensemble (GOE) and at B large enough to break time-reversal (TR) symmetry by the Gaussian unitary ensemble (GUE) [7]. This implies that if we expand $X_\lambda(x,y)=\sum_{\mu=1}^\infty a_\mu\rho_\mu(x,y)$, where $\rho_\mu(x,y)$ is an arbitrary basis, the coefficients $\{a_\mu\}$ should be uniformly distributed in Hilbert space [7]. After truncating the basis to a finite d-dimensional set (since the GOE description fails at lengths less than k_f^{-1}) this means that the joint probability density is

$$P(\{a_\mu\})=\frac{2}{\Omega_d}\delta\left(\sum_{\mu=1}^d|a_\mu|^2-1\right), \qquad (3)$$

where Ω_d is the solid angle in d dimensions. If we define $\eta_{\mu,n}=\int_0^W dy\,\rho_\mu(a,y)\phi_n(y)$ then we have (suppressing the index n), $\tilde{\gamma}=\sum_{\mu=1}^d a_\mu\eta_\mu=\mathbf{a}\cdot\boldsymbol{\eta}$. It follows that

$$P(\tilde{\gamma})=\int\prod_{\mu=1}^d da_\mu\,\delta(\tilde{\gamma}-\mathbf{a}\cdot\boldsymbol{\eta})\frac{2}{\Omega_d}\delta\left(\sum_{\mu=1}^d|a_\mu|^2-1\right). \qquad (4)$$

Since the $\{\rho_\mu\}$ must be complete as $d\to\infty$ we must have $\boldsymbol{\eta}\cdot\boldsymbol{\eta}\propto d$; hence by choosing $\boldsymbol{\eta}\|\mathbf{a}_d$ we obtain

$$P(\tilde{\gamma})\propto\int\prod_1^{d-1}da_\mu\,\delta\left(\sum_{\mu=1}^{d-1}|a_\mu|^2-(1-\tilde{\gamma}^2/d)\right)\sim e^{-\tilde{\gamma}^2/2}, \qquad (5)$$

where we have taken the limit $d\gg 1$. This calculation was sketched for real $\tilde{\gamma}$ describing the GOE case of real wave functions; the same reasoning holds for $\mathrm{Re}[\tilde{\gamma}]$ and $\mathrm{Im}[\tilde{\gamma}]$ in the GUE case which thus have a joint Gaussian distribution in each channel. For the case of M channels per lead it is easily shown that the scaled total width $\Gamma/\bar{\Gamma}$ has a χ_ν^2 distribution with $\nu=2\beta M$ ($\beta=1,2$ for GOE, GUE),

$$P_\nu(\Gamma)=A_\nu\Gamma^{\nu/2-1}\exp(-\nu\Gamma/2\bar{\Gamma}), \qquad (6)$$

where $A_\nu=(\nu/2\bar{\Gamma})^{\nu/2}/\mathcal{G}(\nu/2)$ and $\mathcal{G}(p)$ is the gamma function. The case $\nu=1$ corresponds to the famous Porter-Thomas distribution [6]; it only arises here if $B=0$ and the dot has reflection symmetry implying $\Gamma_N^l=\Gamma_N^r$. In this case $\alpha=\frac{1}{2}(\Gamma/\bar{\Gamma})$ and $\mathcal{P}_1(\alpha)=\chi_1^2(\alpha)$ also has the Porter-Thomas form. We note that $P_\nu(\Gamma)$ is peaked at small widths for $\nu=1,2$ but as ν increases it peaks at a value approaching 1 and its variance decreases. Using Eqs. (1) and (6) we may express $\mathcal{P}_\nu(\alpha)$ as a one-dimensional integral [13] for all ν. The most relevant cases are $\nu=2,4$ which describe an asymmetric dot with a single decay channel per lead at $B=0$, $B\neq 0$, respectively. One finds

$$\mathcal{P}_2(\alpha)=(2/\pi\alpha)^{1/2}e^{-2\alpha}, \qquad (7)$$

$$\mathcal{P}_4(\alpha)=2e^{-4\alpha}\int_0^\infty dz\,e^{-z}\left(\frac{z+4\alpha}{z}\right)^{1/2}. \qquad (8)$$

$\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_4$ are plotted in Fig. 1; note the substantial suppression of small amplitudes caused by breaking TR symmetry. All moments of $\mathcal{P}_\nu$ can be calculated analytically [13] and some interesting results are predicted. First, *breaking TR symmetry reduces amplitude fluctuations:* The variances satisfy $\Delta\alpha_2^2=\frac{1}{8}=0.125$, $\Delta\alpha_4^2=\frac{4}{45}\approx 0.089$. This follows simply from the decrease in the variance of the underlying χ^2 distributions as ν increases. Second, *breaking TR symmetry increases the mean amplitude:* $\bar{\alpha}_2=\frac{1}{4}$, $\bar{\alpha}_4=\frac{1}{3}$. Because it relates to $\bar{\alpha}$, this prediction should hold *even when* $kT\gg\Delta\varepsilon$, i.e., for metallic samples as well. The origin of this effect can be seen qualitatively by expanding each partial width in α for small variations around their mean which yields $\bar{\alpha}\approx\frac{1}{4}[1-\Delta(\Gamma/\bar{\Gamma})^2]$. Since $\Delta(\Gamma/\bar{\Gamma})^2$ decreases when TR symmetry is broken, $\bar{\alpha}$ increases as long as the symmetry-breaking field B_c has negligible effect on $\bar{\Gamma}$ [see in-

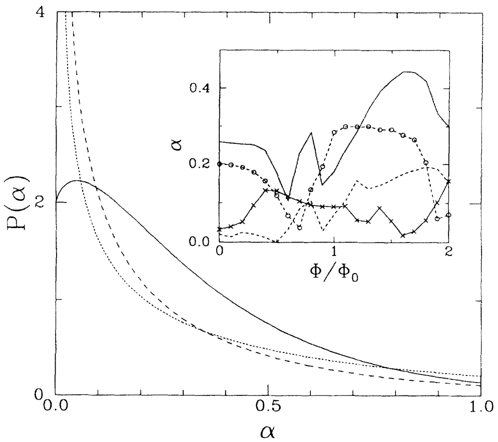

FIG. 1. Distribution $\mathcal{P}_\nu(\alpha)$ for the CB resonance amplitude α for three cases: symmetric dot at $B=0$ [$\nu=1$ (dotted)], asymmetric dot at $B=0$ [$\nu=2$ (dashed)], and asymmetric dot at $B\neq 0$ [$\nu=4$ (solid)]. Inset: Mesoscopic fluctuations of $\alpha/\bar{\alpha}$ vs flux Φ through dot for four consecutive peaks (in energy) for desymmetrized stadium (inset Fig. 2).

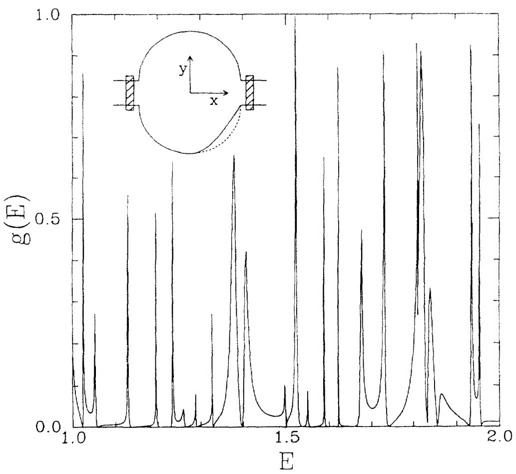

FIG. 2. Conductance $g/(e^2/h)$ vs $E=\varepsilon_F$ for desymmetrized stadium (inset).

set, Fig. 3(a)]. Since $\bar{\Gamma}\ll\Delta\varepsilon$, B_c should be determined for closed chaotic billiards. A plausible extension [13] of recent work on orbital magnetism in disordered systems [14] suggests that $B_cL^2\equiv\Phi_c\sim(h/e)/(\hbar v_F/\Delta\varepsilon L)^{1/2}$, where L is a typical linear dimension of the billiard. The same flux will scramble the amplitude pattern for a single sample and this criterion predicts $B_c\sim 500$ G, consistent with experiment [9]. The numerical results of Figs. 1 and 3(b) (insets) are consistent with this conjecture for B_c but cannot confirm it, as the factor $(\hbar v_F/\Delta\varepsilon L)^{1/2}\sim 1$.

To test the theory microscopically we have performed numerical calculations of conductance resonances (Fig. 2) in a model system consisting of the stadium billiard [11] connected to leads [10] through tunnel barriers of height $15\varepsilon_1$ (ε_1 is the first subband threshold) and desymmetrized by replacing one quarter-circle by a cosine curve (inset, Fig. 2). Although matrix element statistics for chaotic systems have been studied previously with results consistent with Eq. (6) [15], we found no previous direct tests of resonance statistics for chaotic systems. We studied $g(E)$ at fixed B for energies between ε_1 and $2\varepsilon_1$ so that only the single-channel cases ($\nu=2,4$) occur. From Weyl's law [11] this interval should contain roughly the 24th to 49th level of the stadium.

Quantitative statistical analysis requires more than the 25 resonances given by the billiard of Fig. 2 (inset); since in the experimental systems smooth potential fluctuations on the scale of the dot are to be expected anyway, we construct a weakly random ensemble of dots based on this billiard. We add to the "floor" of the dot three randomly located hills (or valleys) smoothly joined to the edges with random height always less than the electron kinetic energy and uniformly distributed in the interval $[-\varepsilon_1, +\varepsilon_1]$; strikingly, this small perturbation produces an uncorrelated resonance pattern. With this procedure we generate resonances for thirty samples and fit the total half-widths to Breit-Wigner line shapes. We find that $\bar{\Gamma}(E)$ varies smoothly by roughly a factor of 2 over the interval studied, and agreement to theory is significantly improved if the resonances are rescaled by a linear $\bar{\Gamma}(E)$ instead of by a constant $\bar{\Gamma}$ [see inset, Fig. 3(a)]. The fitting requires some care and details will be given elsewhere [13]. In the histograms of Fig. 3 we compare the distribution of total widths Γ (corresponding to $T=0$ behavior) to the predictions of Eq. (6) [this is sufficient since the distribution $\mathcal{P}_\nu(\alpha)$ follows analytically]; due to finite energy resolution the correct population of the first bin (small Γ) is not resolved and we need to compare the data to truncated χ^2 distributions [13]. We find excellent agreement for the case $B=0$ and reasonable agreement for B corresponding to flux $\Phi=2\Phi_0$. The crossover scale for TR symmetry breaking is quantified by fitting the distribution at intermediate fields [15] to ν between 2 and 4 [inset, Fig. 3(b)]. Unfortunately no strong saturation at $\nu=4$ (corresponding to GUE) is found; we have numerical evidence [13] that the continued B dependence of $P(\Gamma)$ occurs because the cyclotron radius approaches the dot radius and beyond this field scale $B_{\rm edge}$ the chaotic motion is suppressed by edge-state formation [3]. Roughly $B_{\rm edge}/B_c\sim\sqrt{N}$, so unlike our simulations in typical experimental systems with $\sqrt{N}\sim 10$–25 there should be a large field interval over which our current theory will apply.

In summary, amplitude fluctuations of CB oscillations in semiconductor quantum dots arise from quantum-chaotic fluctuations in the decay widths of quasibound states which have universal distribution laws describable by random-matrix theory. A weak magnetic field enhances the mean amplitude while reducing fluctuations in the amplitude.

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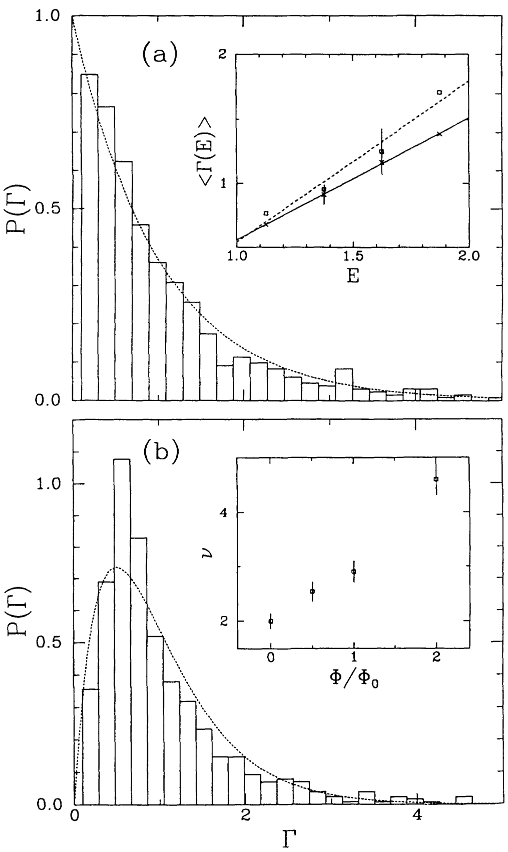

FIG. 3. Histograms of total widths $\Gamma(E)$ normalized by $\bar{\Gamma}(E)$ for (a) $\Phi=0$ and (b) $\Phi/\Phi_0=2$. Dotted curves are expected χ^2_ν distributions of Eq. (6) with (a) $\nu=2$, (b) $\nu=4$. Inset (a): $\bar{\Gamma}(E)$ obtained from numerical data and best linear fit for $\Phi=0$ (squares, dash) and $\Phi/\Phi_0=2$ (crosses, solid). Only typical error bars are shown. Inset (b): Best ν obtained by fitting histograms obtained at several flux values to a truncated χ^2_ν distribution showing field scale for $\nu=2\rightarrow 4$ transition [15].

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