Quantum Noise Reduction in a Spatial Dissipative Structure

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We give the quantum-mechanical formulation of a model which predicts the onset of a spatial dissipative structure in a nonlinear optical system. In the case of roll patterns, we show that the two signal beams which constitute the pattern are correlated twin beams, i.e., their intensity difference exhibits fluctuations below the standard quantum limit.

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Even if ordered spatial structures in dissipative nonlinear systems have been objects of study for a long time (see, e.g., [1,2]), their analysis has remained confined to a purely classical description. On the other hand, optical systems lend themselves naturally to a quantum-mechanical treatment and to the prediction of quantum effects, for example, squeezing [3,4]. In this paper we provide a quantum-mechanical description of the model of nonlinear optical system given in [5], and predict the existence of a purely quantum phenomenon in a spatial, stationary dissipative structure. This work establishes also, for the first time, a link between the field of transverse patterns [6] and of quantum noise reduction [3,4] in nonlinear optical systems, both of which have attracted a lively interest in recent years.

We consider a ring or a Fabry-Pérot cavity of length L containing a medium with a cubic refractive nonlinearity (Fig. 1). A coherent, stationary, plane-wave field E_I with frequency ω_0 is injected into the cavity in the longitudinal z direction. We assume conditions such that only a longitudinal mode of the cavity contributes to the electric field [7], which therefore has the form $E(x,y,t)u(z) \times\exp(-i\omega_0 t)+\text{c.c.}$, where x and y are the transverse coordinates, and $u=\exp(ik_z z)$, $k_z=2\pi n_z/L$ for a ring cavity, $u=\cos(k_z z)$, $k_z=\pi n_z/L$ in the case of a Fabry-Pérot cavity (n_z is a large positive integer). The envelope E obeys the equation [5]

$$\frac{\partial E}{\partial \bar t}=-E+E_I+iE(|E|^2-\theta)+ia\nabla_\perp^2 E\,, \qquad (1)$$

where $\bar t=kt$, with k being the cavity linewidth, and θ is the detuning parameter; we have assumed a self-focusing nonlinearity. In order to avoid difficulties arising from a continuum of transverse modes, we consider in the transverse plane (x,y) a square of side b and we assume periodic conditions for E. Hence E has the form [8]

$$E(x,y,\bar t)=\sum_{\mathbf n} e_{\mathbf n}(\bar t)\exp(i\mathbf k_{\mathbf n}\cdot\mathbf x)\,, \qquad (2)$$

where $\mathbf x\equiv(x,y)$, $\mathbf n\equiv(n_x,n_y)$, $\mathbf k=2\pi\mathbf n/b$, $n_x,n_y=0,\pm1,\pm2,\ldots$. The parameter a and the transverse Laplacian $\nabla_\perp^2$ in Eq. (1) are given by $a=c/2k_z kb^2$, $\nabla_\perp^2=\partial^2/\partial\bar x^2+\partial^2/\partial\bar y^2$, where c is the light velocity, $\bar x=x/b$, and $\bar y=y/b$. By inserting Eq. (2) into Eq. (1), one obtains the following set of equations for the modal amplitudes $e_{\mathbf n}$:

$$\frac{de_{\mathbf n}}{d\bar t}=E_I\delta_{\mathbf n,0}-e_{\mathbf n}(1+i\theta_{|\mathbf n|})+i\sum_{\mathbf n',\mathbf m} e^*_{\mathbf m+\mathbf n'-\mathbf n}e_{\mathbf m}e_{\mathbf n'}\,, \qquad (3)$$

where an asterisk denotes complex conjugation and

$$\theta_{|\mathbf n|}=\theta+\beta_T n^2,\ \ \beta_T=4\pi^2 a,\ \ n^2=n_x^2+n_y^2\,; \qquad (4)$$

the structure of the cubic term in Eq. (3) manifestly preserves the transverse wave vector. Equations (1) and (3) admit transversally homogeneous stationary solutions $e_{\mathbf n}=e_s\delta_{\mathbf n,0}$, which obey the steady-state equation [9]

$$|E_I|^2=|e_s|^2[1+(|e_s|^2-\theta)^2]\,. \qquad (5)$$

The linear stability analysis of this stationary solution, performed in [5], reveals the existence of a steady-state bifurcation which leads to the formation of various stationary patterns, as, for example, rolls (stripes) [5] or hexagons [10]. These structures are closely similar to those found in the case of counterpropagating waves in a cavityless Kerr medium [11].

A quantum-mechanical formulation of the model is provided by the following master equation (ME). Let us

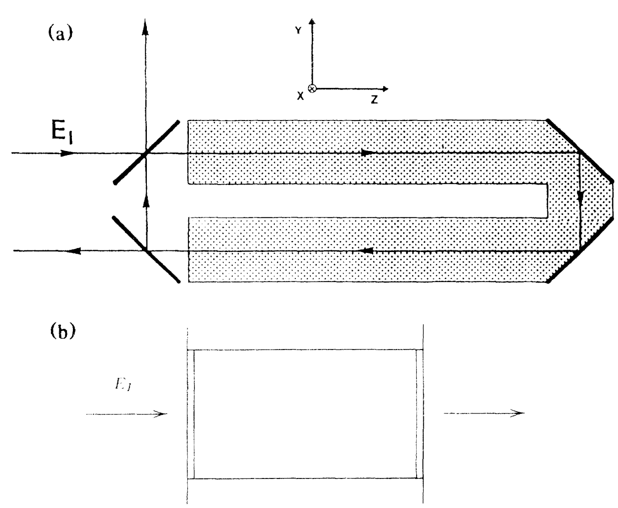

FIG. 1. (a) Ring cavity and (b) Fabry-Pérot cavity.

denote by ρ the density operator of the multimode system, and by $a_{\mathbf{n}}$ and $a_{\mathbf{n}}^\dagger$, the annihilation and creation operators of photons, respectively, for the cavity modes. The ME reads

$$\frac{d\rho}{d\bar{t}}=\sum_{\mathbf{n}}\Lambda_{\mathbf{n}}\rho-\frac{i}{\hbar}[H,\rho]\,, \tag{6}$$

where the Liouvillian $\Lambda_{\mathbf{n}}$ is given by

$$\Lambda_{\mathbf{n}}\rho=[a_{\mathbf{n}}\rho,a_{\mathbf{n}}^\dagger]+[a_{\mathbf{n}},\rho a_{\mathbf{n}}^\dagger]\,, \tag{7}$$

and the Hamiltonian H is the sum of the three contributions

$$H_{\text{ext}}=i\hbar\alpha_I(a_0^\dagger-a_0),\;\; H_0=\sum_{\mathbf{n}}\theta_{|\mathbf{n}|}a_{\mathbf{n}}^\dagger a_{\mathbf{n}}\,, \tag{8a}$$

$$H_{\text{int}}=\left(\frac{\hbar g_0}{2}\right)\int\int dx\,dy[A^\dagger(x,y)]^2A^2(x,y)\,, \tag{8b}$$

where $\alpha_I=E_I/g_0^{1/2}$, and A is proportional to the field envelope

$$A(x,y)=\frac{1}{\sqrt{b}}\sum_{\mathbf{n}}a_{\mathbf{n}}\exp(i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x})\,. \tag{9}$$

The parts H_{ext}, H_0, and H_{int} describe the pump field, the free evolution of the modes, and the mode-mode interaction, respectively. By using Eqs. (6)-(9) one can derive the time evolution equations for the mean values $e_{\mathbf{n}}(\bar{t})\equiv g_0^{1/2}\langle a_{\mathbf{n}}\rangle(\bar{t})$, and one verifies that in the semiclassical approximation (i.e., neglecting the quantum fluctuations and correlations) these equations coincide with Eqs. (3). In the case that Eq. (1) is derived from a two-level model [7], the explicit expression of the coupling constant g_0 is $2C/\Delta^3N_s$, where C, Δ, and N_s are the bistability parameter, the atomic detuning parameter ($\Delta<0$ is the self-focusing case), and the saturation parameter, respectively.

In order to obtain some useful results from the ME, we base ourselves on the link with the problem of oscillatory instabilities in the longitudinal modes of a ring cavity. Precisely, in [12] we considered a ring cavity containing a two-level system (with purely radiative damping), and driven by a coherent field. Contrary to the case of this paper, we assumed that the electric field does not depend on the transverse variables x and y, and that the atomic line excites several longitudinal modes of the cavity; therefore the envelope E depends on z and on the time. In the limit of adiabatic elimination of atomic variables and of large atomic detuning $|\Delta|$, we derived a ME which governs the interaction of the longitudinal modes. The analysis of [12] was limited to three modes; however, the ME obtained there can be generalized in a straightforward way to include all the longitudinal modes, and turns out to coincide with Eqs. (6)-(9), provided the transverse length b is replaced by the longitudinal length L, $\mathbf{x}$ is replaced by z, the two-component index $\mathbf{n}$ is replaced by a one-component index $n=0,\pm1,\ldots,$ and β_T in Eq. (4) is replaced by $\beta_T=8C(\pi c)^2/(-\Delta)(L\gamma_\perp)^2$. The mean values $g_0^{1/2}\langle a_n\rangle(\bar{t})$ obey the same semiclassical equations (3); the steady-state equation (5) and the linear stability analysis remain unchanged. However, in this case the amplitudes of the longitudinal modes are given by $e_n(t)=\exp[-i(2\pi nc/L)t]g_0^{1/2}\langle a_n\rangle$, and therefore the structures which arise in the unstable domain are not stationary but oscillatory, even when $\langle a_n\rangle$ is time independent.

Next, let us come back to the problem of transverse modes, and let us consider the simplest case of rolls, assuming for definition that the roll pattern develops in the x direction. The experimental observation of rolls is reported in [11c]. Close to the critical point $|e_s|=1$ (see Ref. [5]), the dynamics is governed by the pump mode $\mathbf{n}=\mathbf{0}$ and by the two signal modes which become unstable at the critical point, i.e., $\mathbf{n}=(n_c,0)$ and $\mathbf{n}=(-n_c,0)$, where n_c is such that $(2\pi n_c)^2=2-\theta$ (see Ref. [5]). Signal and pump beams are shown in Fig. 2(a) for the case of unidirectional propagation (ring cavity). If we label by 1,2,3 the three modes $\mathbf{n}=\mathbf{0}$, $\mathbf{n}=(n_c,0)$, and $\mathbf{n}=(-n_c,0)$, respectively, and write explicitly the interaction Hamiltonian (8b) for the three modes, we obtain that H is given by the sum of the three contributions

$$H_{\text{SFM}}=\frac{\hbar g_0}{2}\sum_{i=1}^{3}(a_i^\dagger)^2a_i^2\,, \tag{10a}$$

$$H_{\text{CPM}}=\hbar g_0\{a_1^\dagger a_1a_2^\dagger a_2+a_1^\dagger a_1a_3^\dagger a_3+a_2^\dagger a_2a_3^\dagger a_3\}\,, \tag{10b}$$

$$H_{\text{FWM}}=2\hbar g_0\{a_1^2a_2^\dagger a_3^\dagger+\text{H.c.}\}\,, \tag{10c}$$

where H_{SFM}, H_{CPM}, and H_{FWM} describe the processes of self-phase modulation, cross-phase modulation, and four-wave mixing, respectively. The four-wave mixing provides the gain which primes the onset of the dissipative structure. In this problem the two signal beams have the same frequency, contrary to the case of longitudinal modes [12]. The three-mode problem for the ME (6) with the Hamiltonian (8) coincides exactly with that studied in [12]. For example, in [12] we calculated the steady-state equations which describe the multimode stationary solutions.

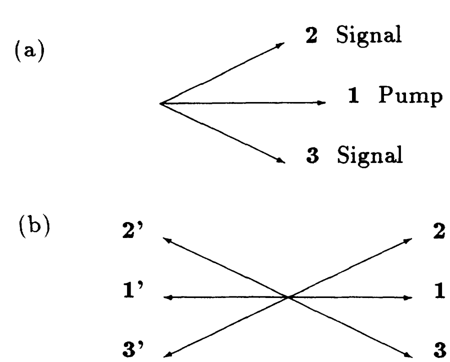

FIG. 2. Pump and signal beams in a roll pattern: (a) ring cavity and (b) Fabry-Pérot cavity.

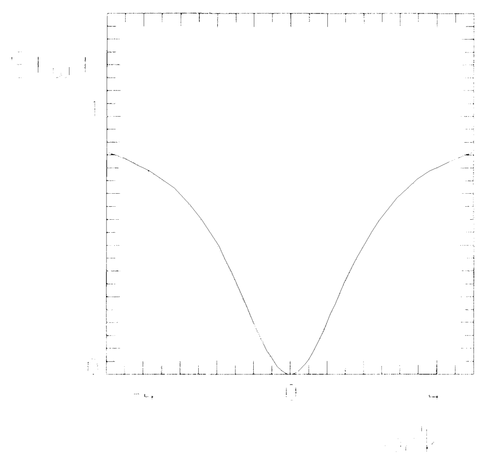

FIG. 3. Spectrum of the quantum fluctuation in the intensity difference between the two signal modes.

Let us now focus on a genuine quantum-mechanical quantity, namely, the spectrum of the fluctuations in the intensity difference between the signal beams of the roll pattern. This quantity has been already calculated in [12] for the corresponding longitudinal three-mode problem. The result is

$$S(\omega)=1-4k^2/(\omega^2+4k^2) \tag{11}$$

and is graphed in Fig. 3. Therefore one has fluctuations below the shot noise level $S=1$ (standard quantum limit), and the reduction of quantum noise in the intensity difference becomes complete for zero frequency [$S(0)=0$]. Equation (11) coincides with the expression of the spectrum of the intensity difference between the signal beams in the optical parametric oscillator [13]. This result demonstrates the quantum correlation between the two beams, which for this reason are usually called "twin beams." In our case, however, the two twin beams propagate along different directions and in the far field the signal beams are spatially well separated (Fig. 2). In the case of a Fabry-Pérot cavity, we must take into account that there are four signal beams, as shown in Fig. 2(b), and that the annihilation operator a_2 (a_3), refers simultaneously to the beams 2 and 2′ (3 and 3′). Therefore the function $S(\omega)$ corresponds to the spectrum of the fluctuations in the difference between the total intensity of the beams $2+2'$ and the total intensity of the beams $3+3'$.

In conclusion, we have demonstrated for the first time the existence of a quantum feature in a spatial dissipative structure. This feature is the quantum-mechanical correlation of the two beams which correspond to a roll pattern, so that they behave as twin beams. One can expect that similar correlations can exist also in the case of more complex patterns as, e.g., hexagons. Therefore the order which characterizes these spatial dissipative structures also affects the quantum-mechanical level. Our results have been obtained by exploiting an "equivalence" between the problem of transverse stationary pattern formation and that of oscillatory instabilities in the longitudinal modes of a ring cavity.

Left for future work is the investigation of some other quantities which may show quantum noise reduction, as, e.g., the fluctuations of the intensities of the two individual signal beams, or of the sum of their intensities, or of the sum of their phases, or of the intensity of the pump mode. Very interesting work also would be to identify quantities, directly related to the near field pattern, which exhibit quantum noise reduction. A possibility might be, for example, the local transverse intensity distribution in the roll pattern.

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[1] G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977).

[2] H. Haken, *Synergetics, an Introduction* (Springer-Verlag, Berlin, 1977).

[3] R. Loudon and P. L. Knight (editors), J. Mod. Opt. **34** (1987).

[4] H. J. Kimble and D. F. Walls (editors), J. Opt. Soc. Am. B **4** (No. 10) (1987).

[5] L. A. Lugiato and L. Lefever, Phys. Rev. Lett. **58**, 2209 (1987).

[6] See, e.g., the special issues on *Transverse Effects in Nonlinear Optical Systems*, edited by N. B. Abraham and W. J. Firth [J. Opt. Soc. Am. B **7** (Nos. 6 and 7) (1990)]; J. V. Moloney, H. Adachiara, D. W. McLaughlin, and A. C. Newell, in *Chaos, Noise and Fractals*, edited by R. Pike and L. A. Lugiato (Hilger, Bristol, 1987).

[7] L. A. Lugiato and C. Oldano, Phys. Rev. A **37**, 3896 (1988); L. A. Lugiato, K. Wang, and N. B. Abraham (to be published).

[8] This is different from Ref. [5], in which one adds to the cavity two lateral mirrors, and one assumes that E is independent on y and obeys Neumann boundary conditions with respect to x. However, the results of the linear stability analysis of the homogeneous stationary solution remain unchanged.

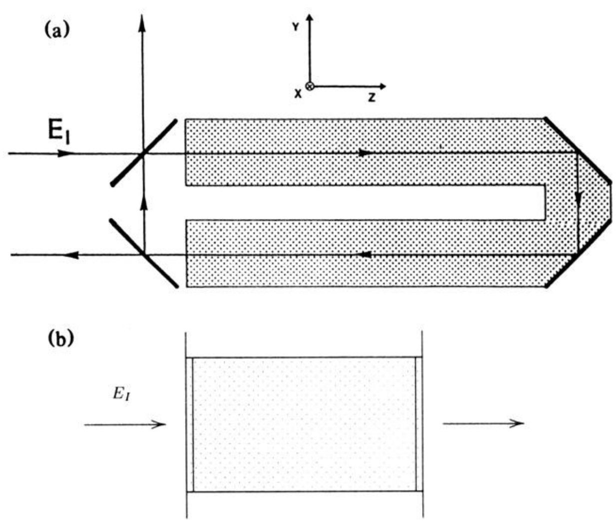
[9] H. M. Gibbs, S. L. McCall, and T. N. C. Venkatesan, Phys. Rev. Lett. **36**, 113 (1976).

[10] W. J. Firth, G. S. McDonald, A. J. Scroggie, and L. A. Lugiato (to be published).

[11] (a) G. Grynberg, E. LeBihan, P. Verkerk, P. Simoneau, J. R. R. Leite, D. Bloch, S. Le Boiteux, and M. Duclay, Opt. Commun. **67**, 363 (1988); (b) W. J. Firth, J. Mod. Opt. **37**, 151 (1990); (c) A. Petrossian, M. Pinard, A. Maitre, J. Y. Courtois, and G. Grynberg (to be published).

[12] M. Brambilla, F. Castelli, L. A. Lugiato, F. Prati, and G. Strini, Opt. Commun. **83**, 363 (1991).

[13] S. Reynaud, C. Fabre, and E. Giacobino, J. Opt. Soc. Am. B **4**, 1520 (1987).



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