Nonlinear 1D Laser Pulse Solitons in a Plasma

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(Received 31 October 1991)

A class of exact one-dimensional solutions for modulated light pulses coupled to electron plasma waves in a relativistic cold plasma is investigated. The solutions are in the form of isolated envelope solitons and the nonlinear relationship between their group velocity, amplitude, and frequency are discussed. Numerical results are presented for intense pulses propagating close to the velocity of light; such pulses are of great interest from the point of view of particle and photon accelerators.

PACS numbers: 52.35.Mw, 52.40.Db, 52.40.Nk, 52.75.Di

Recent developments in laser technology, such as the generation of picosecond pulses with energies up to tens of joules and possible applications to particle accelerators, photon accelerators, etc., have led to a resurgence of interest in the study of intense ($v_{\rm osc}/c > 1$) laser pulse propagation in cold plasmas [1]. One of the problems of key interest from the point of view of particle and photon acceleration is the magnitude of the group velocity of such pulses since it determines the propagation speed of the associated plasma wave wake field. This question has been addressed in the past by computer simulations using particle codes, numerical solution of the relevant nonlinear equations, etc. An attempt has also been made [2] to estimate the group velocity from the nonlinear dispersion relation for exact solutions [3–5] of relativistically intense circularly polarized waves. However, since these well-known exact solutions are for infinite unmodulated nonlinear waves, it seems inappropriate to use them for the estimate of nonlinear group velocity for short pulses. In this Letter, we examine a class of exact one-dimensional nonlinear solutions of the relativistic cold plasma equations which represent envelope solitons of light waves [6,7], in which the modulation envelope propagates as a large amplitude plasma wave in the medium. These solutions are a step beyond the well-known stationary solutions in a cold plasma because the envelope and the phase propagate with different speeds so that the nonlinear relationships between the phase and group velocities can be investigated. For simplicity and clarity, we restrict our analysis to circularly polarized waves which couple to longitudinal disturbances only because the amplitude is modulated. These intense light pulses might form interesting candidates for photon accelerator schemes [8] in which the combined effect of the change in plasma density and effective electron mass (due to the relativistic effect) may be used to get significant improvement in the frequency multiplication that is possible.

We start with the well-known relativistic set of fluid equations for a cold plasma (in one dimension) [1,3–5]:

$$\frac{\partial n}{\partial t}+\frac{\partial(nu)}{\partial x}=0\,, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)(\gamma u)=\frac{\partial\phi}{\partial x}-\frac{1}{2\gamma}\frac{\partial A^2}{\partial x}\,, \tag{2}$$

where $\gamma=[(1+A^2)/(1-u^2)]^{1/2}$, u is the longitudinal velocity, and the transverse velocity is eliminated by the exact integration $\mathbf{u}_\perp=\mathbf{A}/\gamma$. The normalizations are n/n_0, u/c, eA/mc^2, $e\phi/mc^2$, $\omega_p t$, $\omega_p x/c$, etc. We now introduce a change of variables $x-\beta t=\xi$, $t=\tau$ and make the key ansatz that in the moving frame the vector potential is circularly polarized and has a sinusoidal phase variation, i.e., $\mathbf{A}=[\{\hat{\mathbf{y}}a(\xi)+\hat{\mathbf{z}}ia(\xi)\}\exp(-i\lambda\tau)+\text{c.c}]$. The introduction of a frequency parameter λ in the phase factor is the basic deviation from earlier stationary wave solutions [3–5]; this allows us to identify β with the group velocity of the light pulse and also to distinguish between the group and phase speeds. The choice of circular polarization allows us to avoid the generation of harmonics in all wave fields. For plasma oscillations, which form the modulation envelope, this leads to $\partial/\partial\tau=0$ in the moving frame. Integrating the fluid equations we then get $n(\beta-u)=\beta$ and $\gamma(1-\beta u)-\phi=1$. Using these relations and further writing $a(\xi)=R\exp(i\theta)$, we can reduce the Poisson's equation and the wave equation to obtain the following set of coupled nonlinear equations:

$$\phi''=u/(\beta-u)\,, \tag{3}$$

with u given by

$$u=\frac{\beta(1+R^2)-(1+\phi)[(1+\phi)^2-(1-\beta^2)(1+R^2)]^{1/2}}{(1+\phi)^2+\beta^2(1+R^2)} \tag{4}$$

and

$$R''+\frac{R}{1-\beta^2}\left[\left(\lambda^2-\frac{M^2}{R^4}\right)\frac{1}{1-\beta^2}-\frac{\beta}{\beta-u}\frac{1-\beta u}{1+\phi}\right]=0\,, \tag{5}$$

where $M=R^2\{(1-\beta^2)\theta'-\lambda\beta\}$ is a constant of integration. In our representation the amplitude $R=(A_y^2+A_z^2)^{1/2}$ of the circularly polarized electromagnetic wave exhibits a modulation propagating at the group speed β.

The question of the definition of phase speed is a little more complex. For small amplitudes R is always positive and the exponential factor contains the entire phase information. The phase speed may then be shown to be simply $1/\beta$ satisfying the conventional relation $V_p V_g = 1$. However, for arbitrary amplitudes R oscillates because of the strong $R-\phi$ coupling and the phase speed has to be defined by directly determining the z,t variation of $R\cos(\theta-\lambda\tau)$.

Equations (3)–(5) are the final set of coupled equations that we must solve for studying the propagation of modulated light pulses in a cold plasma. These equations contain one exact integral of motion which may be written as

$$K=\frac{R'^2}{2}-\frac{\phi'^2}{2(1-\beta^2)}+V(R,\phi)\,, \tag{6}$$

where

$$V(R,\phi)=\frac{\lambda^2}{(1-\beta^2)^2}\frac{R^2}{2}+\frac{M^2}{2R^2(1-\beta^2)^2}-\frac{\phi}{1-\beta^2}-\frac{\beta}{(1-\beta^2)^2}[\beta(1+\phi)-\{(1+\phi)^2-(1-\beta^2)(1+R^2)\}^{1/2}]\,. \tag{7}$$

Since $\beta<1$ (group velocity less than c), the problem is similar to a Hamiltonian of coupled anharmonic oscillators with 2 degrees of freedom [9] (R,ϕ) where the effective mass for one of the anharmonic oscillators is negative. In the limit of weak density response, the problem can be made one dimensional by the substitutions $\phi\approx(1+R^2)^{1/2}-1$ and $n\approx(1+R^2)^{1/2\prime\prime}$. Expanding in R^2 and taking $M=0$, $K=0$ we get the well-known envelope soliton solution

$$A_y=R_m\,\mathrm{sech}\left[\frac{(1-\beta^2-\lambda^2)^{1/2}}{1-\beta^2}(x-\beta t)\right]\times\cos\left[\frac{\lambda\beta}{1-\beta^2}\left(x-\frac{t}{\beta}\right)\right], \tag{8}$$

where

$$R_m=4\left[\frac{(1-\beta^2)(1-\beta^2-\lambda^2)}{4\lambda^2-(1-\beta^2)(3+\beta^2)}\right]^{1/2} \tag{9}$$

$(K\neq 0)$ gives modulated periodic wave-train solutions with the envelope factor described by an elliptic function. Equation (8) represents an isolated light pulse with a frequency $\omega=\lambda/(1-\beta^2)$, wave number $k=\lambda\beta/(1-\beta^2)$, phase velocity $1/\beta$, group velocity β (note $V_pV_g=c^2$), an envelope scale length (in units of c/ω_p) equal to $(1-\beta^2)/(1-\beta^2-\lambda^2)^{1/2}$, and an amplitude-group-velocity relationship given by Eq. (10). Note that the envelope scale length is real only when $\lambda^2<1-\beta^2$. This may be physically interpreted as follows. The wave frequency in the frame moving with the group velocity has the Doppler-shifted value $\bar{\omega}=(\omega-k\beta)/(1-\beta^2)^{1/2}=\lambda/(1-\beta^2)^{1/2}$. Thus the inequality $\lambda<(1-\beta^2)^{1/2}$ corresponds to the situation when the Doppler-shifted wave frequency $\bar{\omega}/\omega_p<1$, i.e., the electromagnetic wave finds itself in an overdense plasma and is totally trapped; this is why we get a soliton solution and the wave does not leak out. The effective scale length for trapping may now be seen to be $(1-\beta^2)^{1/2}c/(\omega_p^2-\bar{\omega}^2)^{1/2}$ which is quite reasonable. Similarly the amplitude–group-velocity relationship may be written in terms of physical parameters as

$$1-\beta^2=\frac{-(R_m^2-4+4R_m^2\omega^2)+[(R_m^2-4+4R_m^2\omega^2)^2+64R_m^2\omega^2]^{1/2}}{8\omega^2}\,. \tag{10}$$

For arbitrary amplitudes we need to analyze the fully two-dimensional problem [Eq. (7)] in which the nonlinear potential function has a complicated structure. Our primary interest is to investigate soliton solutions for which we have solved Eqs. (3)–(5) numerically and looked for solutions that decay exponentially as $\xi\to\pm\infty$. For λ very close to $(1-\beta^2)^{1/2}$ we get small-amplitude solitons which are well described by the analytic solutions (8) and (9). As λ is decreased the soliton amplitude increases and acceptable solutions occur only at discrete values of λ. In other words, for a fixed value of β finding soliton solutions turns out to be an eigenvalue problem in λ. The sizes and shapes of these solitons also vary as a function of λ. Typically ϕ has a characteristic bell shape whereas R has a number of nodes. Figure 1 shows a typical soliton solution for $\beta=0.97$ and $\lambda=0.224445$. For applications such as particle acceleration or photon acceleration, the regime of interest is $\beta\to 1$, where the group velocity is close to c. We have carried out a detailed investigation of soliton solutions in this regime. Figure 2 is a plot of normalized group velocity V_g versus the normalized carrier frequency Ω $(=\omega/\omega_p)$. The solid curve corresponds to soliton pulse results obtained from our numerical work. For comparison we have also plotted the linear group velocity (dashed line) $V_{gL}=(1-1/\Omega^2)^{1/2}$ and the nonlinear group velocity for the infinite plane wave [3–5] (dotted line), $V_{g\infty}=\beta[1-(\gamma_\infty-1)/2\Omega^2\gamma_\infty(\gamma_\infty+1)]$. This last expression has been recently obtained by Mori *et al.* [2] and depends on the infinite wave γ, viz., $\gamma_\infty=(1+A^2)^{1/2}$. The nonlinear group velocities are closer to c than the linear group velocity. This is physically understandable because the nonlinearity makes the electrons heavier and thereby weakens the plasma dielectric effects. Finite width soliton pulses propagate slower than the infinite plane waves. Physically this is because coupling to plasma waves acts as a drag on the electromagnetic waves

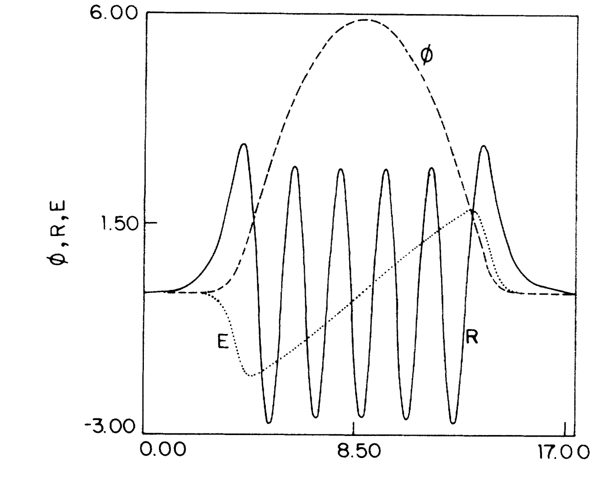

FIG. 1. Electrostatic potential ϕ (dashed line), electric field E ($=-\partial\phi/\partial\xi$, dotted line), and electromagnetic wave amplitude R (solid line) profiles for a soliton pulse with $p=6$, $\beta=0.97$, and $\lambda=0.224445$.

and slows them down. The deviations between $V_{g\infty}$ and V_{gs} can be substantial; in our numerical work, we have observed (not shown) deviations up to 25% of the difference between V_g and 1.

As stated earlier, the soliton pulses can only be obtained for characteristic combinations of amplitude R_0,

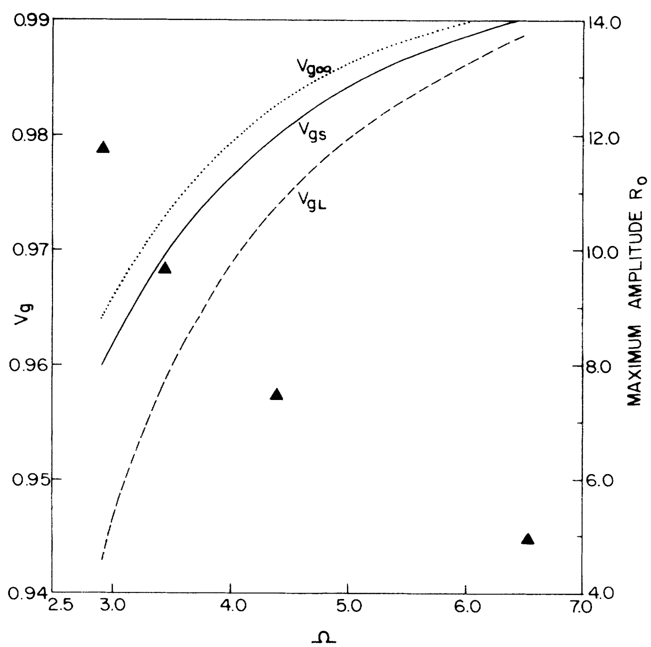

FIG. 2. Normalized group velocity vs normalized frequency curves for soliton pulses (solid line), infinite pulses (dotted line), and linear pulses (dashed line). The maximum electromagnetic wave amplitude R_0 for the soliton pulse (for $p=18$) is indicated by solid triangles.

group velocity $\beta=V_{gs}$, and frequency Ω. Figure 2 also shows the values of R_0 (solid triangles) for which the soliton pulse solutions are obtained. Note that in the V_{gs}-Ω curves, the changing R_0 and Ω are working against each other. Increasing Ω increases the group velocity but the simultaneously decreasing value of R_0 opposes this tendency and we see the resulting effect in our plots. This point is emphatically brought out in Table I where we show values of R_0 and Ω for which there is an exact compensation of these two opposing tendencies such that the group velocity V_{gs} is constant.

To understand these solutions better we consider another approximate analytic limit to Eqs. (3)-(5) particularly applicable when the ϕ soliton structure is large. When $\phi_{\max}$ is large, the longitudinal velocity $u\to-1$, $n\to\beta(1+\beta)^{-1}$, and $n/\gamma\to 0$. In this case, the light wave propagates essentially in a plasma-free region (because the residual electrons have become infinitely massive) and we can approximate the effective Hamiltonian as

$$K_{\text{eff}}=\frac{R'^2}{2}+\frac{\lambda^2}{(1-\beta^2)^2}\frac{R^2}{2}-\frac{\phi'^2}{2(1-\beta)^2}-\frac{\phi}{(1-\beta^2)(1+\beta)}+\frac{\beta}{(1+\beta)(1-\beta^2)}+\cdots, \tag{11}$$

where the ellipses refer to terms of order $\leq\phi^{-1}$ which are neglected. From (11) we get $R=R_0\sin[\lambda\xi/(1-\beta^2)]$. Furthermore, the structure of ϕ near its maximum can be approximated by the parabolic equation [$u\to-1$ in Eq. (3)] $\phi=\phi_{\max}-\xi^2/2(1+\beta)$, which gives an estimate of the spatial scale size of the soliton as $\xi_{\max}\simeq[2(1+\beta)\phi_{\max}]^{1/2}$. From Eqs. (6) and (7) we know that $K=0$ for a soliton solution. Noting that $K\simeq K_{\text{eff}}$ for $\phi=\phi_{\max}$, $\phi'=0$, we get $\phi_{\max}=\beta+\lambda^2R_0^2/2(1-\beta)$. Furthermore, if the soliton region involves p wavelengths of the light wave, we have $p\pi(1-\beta^2)/\lambda\simeq\xi_{\max}$. The above expressions lead to the following approximate analytic relationship between λ, R_0, and β:

$$R_0^2=\frac{2(1-\beta)}{\lambda^2}\left[\frac{\pi^2p^2}{2\lambda^2}(1-\beta^2)(1-\beta)-\beta\right]. \tag{12}$$

Equation (12) is largely corroborated by the numerical work shown in Figs. 1 and 2. Physically these solutions correspond to a soliton in which standing light waves are set up in an "effectively empty" cavity, because $n/\gamma\to 0$ in most of the region. Finally, the n/γ can be made to suffer a change from 1 to a large number before it drops down to zero by making the laser pulse long. In this case, all the density swept out of the pulse piles up in layers

TABLE I. Characteristic frequency (Ω) and soliton amplitude (R_0) for $V_{gs}=0.96$.

Ω	3.3866	3.2391	3.1669	3.0139	2.9460
R_0	1.423	3.216	4.508	8.730	11.760

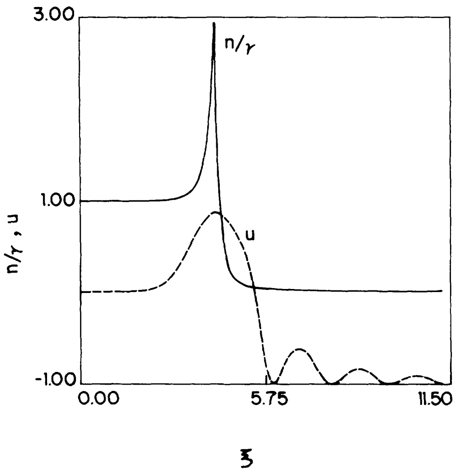

FIG. 3. Variation of $\delta n/\gamma$ (solid line) and u (dashed line) for a soliton pulse with $\beta=0.9$, $\lambda=0.336$, and $p=19$.

with a width of order c/ω_p, which is the skin depth, at either edge of the pulse. Figure 3 illustrates this effect for a $p=19$ soliton with $\beta=0.9$ and $\lambda=0.336$.

To conclude, we have analyzed exact one-dimensional solutions for modulated light pulses coupled to electron plasma waves in a cold plasma. The one-dimensional approximation is reasonable as long as transverse scale lengths are much longer than longitudinal pulse dimensions (which is a few skin depths c/ω_p). Solutions may be in the form of isolated envelope solitons or modulated periodic wave trains. Physically, the soliton pulse may be viewed as a light wave which is trapped in a plasma wave that it generates itself. The front of the pulse generates the plasma wave as a wake field, which is then reabsorbed by the tail of the pulse. The exchange of energy between the light wave and the plasma wave also leads to a "chirping" of the pulse (cf. Fig. 1). This suggests that to experimentally create such pulses, one must not only use characteristic values of R_0 and ω but also do some appropriate chirping of the light wave. The nonlinear relationship between the group velocity and phase velocity and between the group velocity and amplitude, frequency, etc., has also been discussed. Numerical results have been presented for the first time for intense pulses propagating close to the velocity of light; such pulses are of great interest from the point of view of particle and photon accelerators. We have also demonstrated the existence of light pulses where the change in n/γ inside the pulse is well above unity (Fig. 3). One can hope to get a very large frequency multiplication factor from using such pulses as photon accelerators. Thus, taking $\omega_p^2/\gamma=(\omega_p^2/\gamma)_0 f(x)$ where $f(x)$ varies between $f_{\max}$ and $f_{\min}$, and proceeding as do Wilks *et al.* [8], one gets

$$\frac{\omega_f}{\omega_i}=\frac{1+\beta(1-f_{\min}/f_{\max})^{1/2}}{1-\beta(1-f_{\min}/f_{\max})^{1/2}}.$$

For a pulse with $f_{\min}/f_{\max}\to 0$ (see Fig. 3), $\omega_f/\omega_i \to (1+\beta)/(1-\beta)$. For the example shown this ratio is about 20, which is a large factor indeed. Of course one has to worry whether wave breaking or trapping of background plasma electrons can seriously jeopardize the integrity of these solutions or significantly damp them. The photon acceleration time is $\approx L_d/c$ where L_d is the structure length for the sharp n/γ spike whereas the damping time due to trapped electrons is $\approx L_s/c$ where L_s is the size of the Φ soliton and c is roughly the velocity to which the trapped electrons are accelerated. Since $L_d \ll L_s$ in these solutions, photon acceleration may be completed long before the trapped particle effects come in. This conclusion needs further investigation by simulations or experiments.

One of us (P.K.K.) would like to thank C. Joshi and J. Dawson of UCLA for hospitality during the period when this work was initiated. A.S. is grateful to Professor Abdus Salam, the IAEA, UNESCO, and SAREC for hospitality at the ICTP, Trieste, where a part of this work was carried out.

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