Precursory Singularities in Spherical Gravitational Collapse

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General conditions are developed for the formation of naked precursory ("shell-focusing") singularities in spherical gravitational collapse. These singularities owe their nakedness to the fact that the gravitational potential fails to be single valued prior to the onset of a true gravitational singularity. It is argued that they do not violate the spirit of cosmic censorship. Rather, they may well be an essentially generic feature of relativistic gravitational collapse.

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What are the generic features of spherical gravitational collapse? Perhaps one of the most surprising aspects of classical general relativity is the fact that we still do not have a definitive answer. One would hope that the dynamics of the situation is now well understood. This is in fact not the case. The cosmic censorship hypothesis which, in very general terms, asserts that singularities which develop from regular initial conditions have no causal influence on spacetime has yet to be formulated [1]. Indeed, there is a body of evidence which suggests that the hypothesis may not hold [2]. Failure of the cosmic censorship hypothesis would be a disaster for classical general relativity in particular and for physics in general since it would indicate that physical laws may have only limited applicability. Recent numerical [3] and analytic [4] studies have shown that possibly naked singularities can also arise in nonspherical collapse.

The purpose of this Letter is to examine the general conditions under which a naked "shell-focusing" singularity develops in spherical gravitational collapse. No integration of the Einstein field equations is performed. Rather, it turns out that only the most general functional properties of the spacetime are needed. It is shown that the development of naked shell-focusing singularities can be traced to the breakdown of the gravitational potential which becomes multivalued prior to the onset of a true gravitational singularity which *is* clothed in an event horizon. It is argued that this development can be viewed as an essentially generic feature of spherical gravitational collapse, and that it does *not* represent a violation of the spirit of cosmic censorship.

Figure 1 represents a typical spherical gravitational collapse in terms of advanced Bondi coordinates (v,r) (see below). The boundary of the object Σ collapses to zero volume at C. A signifies the onset of a "singularity" at the center of symmetry. The curve AB traps photons as shown, and evolves into the Schwarzschild horizon to the exterior of Σ which we take to be vacuum.

We use advanced Bondi coordinates [5] (v,r,θ,ϕ) so

$$ds^2=2c\,dv\,dr-c^2(1-2m/r)dv^2+r^2d\Omega^2, \qquad (1)$$

where $c=c(v,r)>0, m=m(v,r)$ which we take to be ≥ 0, and $d\Omega^2\equiv d\theta^2+\sin^2\theta d\phi^2$. Note that m can be invariantly defined as the effective gravitational mass [6]. Radial ingoing null geodesics have 4-tangents $k^a=-\delta^a_r/c$. We use the coordinate freedom in v to set $c(v,0)=1$. (This choice is inconsistent with the "self-similar" case, which is exceptional; see below.) In order to examine the regularity conditions, and to distinguish "singularities," we make use of the Kretschmann invariant

$$K\equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}, \qquad (2)$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor. [It is

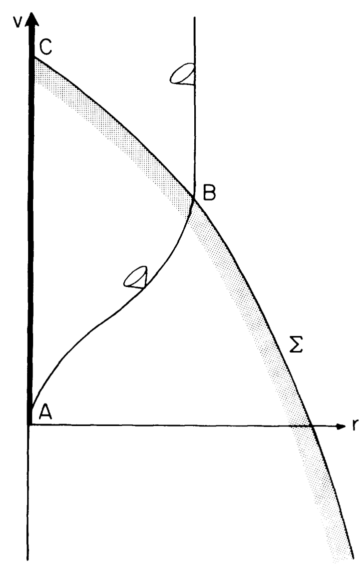

FIG. 1. Two-dimensional picture of the spherical gravitational collapse of a bounded object (with boundary Σ) to zero volume of C. A "singularity" at the center of symmetry begins at A $(v=r=0)$. The apparent horizon in the matter AB $[r=2m(v,r)]$ is shown schematically. Two null cones are shown, in the matter and in the vacuum exterior on the Schwarzschild horizon. The interior event horizon is the outgoing null geodesic from $(v\leq 0, r=0)$ to B. It evolves to the past of AB which we assume is the only solution to $r=2m(v,r)>0$ within the interior. If the event horizon originates from $v<0$ then A is at most locally naked. If it originates from $v=0$ then A can be globally naked.

necessary to make use of computer algebra to manipulate K. Whereas K simply reduces to the familiar $48m^2/r^6$ for the Vaidya metric [2], generally K expands out to 49 distinct terms [7] for the metric (1).]

In order to examine regularity at the origin for ($v<0$, $r=0$) it is convenient to define the function $n(v,r)$ given by

$$m(v,r)\equiv n(v,r)r^{\alpha} \tag{3}$$

and the function $l(v,r)$ given by

$$c(v,r)\equiv \exp\{l(v,r)r^2/2\}\,. \tag{4}$$

If $\alpha<1$ then $\partial m/\partial r$ fails to be defined at $r=0$ for any v. For $1\le\alpha<3$ it follows that K diverges like

$$K\sim 4n^2(v,0)r^{2(\alpha-3)}\{\alpha^4-6\alpha^3-17\alpha^2-20\alpha+12\} \tag{5}$$

as $r\to 0$. For $\alpha=3$, K evolves like

$$K=12\{8n^2(v,0)-4l(v,0)n(v,0)+l^2(v,0)\} \tag{6}$$

along $r=0$ for $v<0$, and for $\alpha>3$ we obtain (6) with $n(v,0)$ set to zero. The vector field $u^{\alpha}=\delta^{\alpha}_{v}$ is timelike for $r^{\alpha-1}<1/2n(v,r)$. Moreover, the acceleration, expansion, and shear vanish along ($v<0$, $r=0$) so that with a decomposition of the energy-momentum tensor of the form

$$T^{\alpha}_{\beta}=(\rho+p)u^{\alpha}u_{\beta}+p\delta^{\alpha}_{\beta}+q^{\alpha}u_{\beta}+q_{\beta}u^{\alpha}+\mathcal{G}k^{\alpha}k_{\beta}$$
$$+\mathcal{H}(l^{\alpha}k_{\beta}+k^{\alpha}l_{\beta})+\mathcal{I}l^{\alpha}l_{\beta}\,, \tag{7}$$

where $l^{\alpha}=\delta^{\alpha}_{r}/2$, $q_{\alpha}=\mathcal{F}\delta^{r}_{\alpha}$, and $\mathcal{F}=-\beta\{\partial T/\partial v+\partial T/\partial r\}$ (T is the temperature and β is the thermal conductivity) we have the following interpretations of $n(v,0)$ and $l(v,0)$ for $\alpha=3$:

$$2\{l(v,0)-3n(v,0)\}=8\pi p\,,$$
$$6n(v,0)=8\pi\{\rho+4\mathcal{G}-2\mathcal{F}\}\,, \tag{8}$$

where $\mathcal{I}=4(\mathcal{G}-\mathcal{F})=2(\mathcal{H}-\mathcal{F})$. For $\alpha>3$ we obtain $2(\mathcal{F}-2\mathcal{G})=-2\mathcal{H}=\rho$ and $2l(v,0)=8\pi p$.

From the metric (1) it follows that in addition to $v=$const radial null trajectories also satisfy

$$\frac{dv}{dr}=\frac{2}{c}\left(1-\frac{2m}{r}\right)^{-1}. \tag{9}$$

This equation will, in general, have "singular points" [8]. Whereas c can fail to be single valued (see below), it is an ambiguity in the "potential" m/r itself which will give rise to an isolated critical point (shell-focusing singularity) in more generic situations. With $m(v<0,r=0)=0$ and $m(v>0,r=0)>0$ (which we will assume) the potential jumps from 0 to infinity creating an isolated critical point for (9) at $v=r=0$. Note that this critical point is in fact a *null surface.* This can be visualized with a standard inversion map $A:\{(v,r)|v=r=0\}\to\{(u,r)|u\neq 0,r=0\}$, where $u\equiv v/r$ and $\{(v,r)|v\neq 0,r=0\}\to\{(u,r)|u=r=0\}$.

Self-similar collapse.—Self-similar spacetimes admit homothetic Killing vectors ξ^{α} where (in local coordinates) $\nabla_{(\alpha}\xi_{\beta)}=(\nabla_{\delta}\xi^{\delta})g_{\alpha\beta}$. Despite the paucity of self-similar spacetimes which represent solutions to Einstein's equations [9], they form the bulk of examples in the literature on shell-focusing singularities [2]. From the metric (1) it follows that without loss in generality we can take $\xi^{\alpha}=(v,r,0,0)$. As a result, $m/r=h(w)/w$ and $c=i(w)$, where $w\equiv r/v$. Here we are interested in the region $w\ge 0$. Note that both c and m/r in general fail to be single valued at A. The homothetic trajectories ($w=$const) are spacelike, null, or timelike for $h(w)>,=,$ or $<w/2-w^2i(w)\equiv j(w)$, respectively. Given $h(w)$ and $i(w)$, the character of the solution is easily found. Intersection of h and $w/2$ gives the values of w associated with the apparent horizon and intersection of h with j gives the values of w associated with the homothetic null geodesic trajectories $w_{\mathcal{N}}$. The $w_{\mathcal{N}}$ are given explicitly in terms of an affine parameter λ by $r=\exp(\lambda)$ for $\kappa=0$ and $r=|\lambda|^{1/\kappa}$ for $\kappa\neq 0$, where [10]

$$\kappa\equiv\left[\frac{w}{i}\frac{di}{dw}-\frac{i}{w}\left(\frac{1}{2}-\frac{dh}{dw}\right)\right]\Bigg|_{w_{\mathcal{N}}}. \tag{10}$$

It follows that given finite $i(w)$ and $h(w)$ any null geodesic from A which evolves through a region of spacelike (timelike) w necessarily evolves to the future (past) of an $w_{\mathcal{N}}$ from A. A is then a topological node [11]. Moreover if there is no $w_{\mathcal{N}}$, A is not visible. If there is one $w_{\mathcal{N}}$ then this is the only null geodesic from A without a turning point. With two $w_{\mathcal{N}}$, nonhomothetic null geodesics approach the smaller $w_{\mathcal{N}}$ as $r\to 0$ and all evolve from A without a turning point. Whereas it is possible to have more than two $w_{\mathcal{N}}$, the character of A is clear. With $\kappa>0$, K diverges at A and the null geodesics which terminate there terminate in a "strong-curvature" singularity [10].

General collapse.—To begin we observe that the radial null geodesics associated with (9) can be given in terms of the autonomous system

$$\dot v=2r,\quad \dot r=c(r-2m)\,, \tag{11}$$

where the overdot denotes $d/dt=B(\lambda)d/d\lambda$ with $c\partial(B/c)/\partial v=\partial[c(r-2m)]/\partial r$ and λ affine. Note that for the geodesics which reach A in the past (where we set $\lambda=0$) $B\sim(1-2\partial m/\partial r)\lambda$. As a result, the singularity at A is at infinite redshift. The formal loss of predictability is mollified by the fact that no information can come out. Letting l^{α} be the associated 4-tangent we obtain

$$\Psi\equiv\lim_{\lambda\to 0}\lambda^2R_{\alpha\beta}l^{\alpha}l^{\beta}=\frac{8\,\partial m/\partial v}{(1-2\,\partial m/\partial r)^2}\Bigg|_A, \tag{12}$$

where $R_{\alpha\beta}$ is the Ricci tensor. The limit Ψ is used to classify the strength of a singularity at A, $\Psi\neq 0$ being a sufficient condition for a "strong" singularity [12]. Equation (12) is remarkable for its simplicity [13]. We now assume that $c(v,r)$ and $m(v,r)$ are expandable about A on v and $r \geq 0$ and set $c(v,0)=1$ and $m(0,0)=0$. It follows that

$$K=\frac{16m_1^2}{r^4}+\frac{48m_2^2u^2}{r^4}+\frac{16m_2c_1(1+u)}{r^3}+\frac{16}{r^2}\{m_4(2m_4-c_1)+c_1(c_1/2+m_3)+2(m_3m_4+c_1m_5)u+3m_5^2u^4\}, \quad (13)$$

where I have used the notation

$$\left(\frac{\partial m}{\partial r},\frac{\partial m}{\partial v},\frac{\partial^2 m}{\partial v\,\partial r},\frac{1}{2}\frac{\partial^2 m}{\partial r^2},\frac{1}{2}\frac{\partial^2 m}{\partial v^2},\frac{\partial c}{\partial r}\right)_A=(m_1,m_2,m_3,m_4,m_5,c_1) \quad (14)$$

and retained $u=(v/r)_A$. Up to second order then all trajectories which reach A reach a singularity (in particular, a scalar polynomial singularity [12]). The singularity is strong for $m_2 \neq 0$.

Before we examine the system (11), note that every null geodesic which evolves from A evolves to the past of the local "apparent horizon" $[r=2m(v,r)]$ at $A[dv/dr|_A \leq (1-2m_1)/2m_2]$. The apparent horizon has no local maximum in v for $\partial m/\partial v < 1/2$. The geodesics can intersect $r=2m(v,r)$ subsequently and so pass through a maximum r (for $\partial m/\partial v > 0$). A is then only *locally* naked. Global nakedness can sometimes be achieved by a suitable choice of boundary Σ (the Cauchy horizon must precede the event horizon within Σ).

With $x \equiv r$ and $y \equiv v$, the matrix of *linear* factors associated with the system (11) is [11]

$$\begin{pmatrix} 1-2m_1 & -2m_2 \\ 2 & 0 \end{pmatrix}. \quad (15)$$

Here we will assume that $m_1 < 1/2$. Extension of (3) to $v=0$ with $\alpha \geq 3$ would argue that *"generic" collapse is characterized by* $m_2=0$. If $m_2=0$ then A becomes a *nonhyperbolic* critical point of the system (11) [14]. The exceptional critical direction (local Cauchy horizon for A) is given by $dv/dr|_A=2/(1-2m_1)$. The remaining geodesics leave A with a vertical tangent. A is a *saddle node,* the union of one parabolic and two hyperbolic sectors in the extended v-r plane. As long as $m_5 > 0$ it follows that the hyperbolic sectors do not contain a segment of v and $r > 0$. As a result, A is at least locally naked for $m_5 > 0$. A is singular [according to (13)] though not strong [according to (12)]. If $m_2 \neq 0$, A is a *hyperbolic* critical point of the system (11) [11]. There are at most two critical directions, with slopes

$$\{(1-2m_1) \pm [(1-2m_1)^2-16m_2]^{1/2}\}/4m_2 .$$

The smaller one distinguishes the local Cauchy horizon. With $m_2 \neq 0$, A is an (unstable) node for $0 < m_2 \leq (1-2m_1)^2/16$. A is at least locally naked unless $m_2 > (1-2m_1)^2/16$, in which case A is a focus. [If $m_2 < 0$, A becomes a saddle, but $m_2 < 0$ is not consistent with the evolution of $m(v,r)$.] In the range $0 < m_2 \leq (1-2m_1)^2/16$ A is singular [according to (13)] and strong [according to (12)]. It is worth noting that energy conditions are *not* in general violated in the range $0 \leq m_2 \leq (1-2m_1^2)/16$ [15].

The foregoing analysis shows that if the mass evolves as $m(v<0,0)=0$ [in particular, m given by (3) with $\alpha \geq 3$], $m(0,0)=0$, and $m(v>0,0)>0$ then as long as $m_2 \not> (1-2m_1)^2/16$ a regular origin evolves into a null singularity at $v=r=0$, which is at least locally naked, before the onset of a massive singularity at ($v>0$, $r=0$), which is causally disconnected from the spacetime. The singularity at $v=r=0$ may well be an essentially generic feature of relativistic gravitational collapse [16]. However, this type of singularity does not violate the spirit of cosmic censorship. It is not that the metric remains bounded (though not single valued) at $v=r=0$, rather it is simply that $m(0,0)=0$. Massless singularities should not be considered gravitational (they do not attract particles), and it is gravitational singularities to which the spirit of cosmic censorship refers. Indeed, as long as $m \geq 0$ it follows directly from (9) that *no spherically symmetric spacetime can evolve a naked gravitational singularity.* Neither geodesic incompleteness, the divergence of K, nor the classification of strength distinguish massless singularities from massive ones. Such a distinction should be considered an essential part of the formulation of the cosmic censorship hypothesis. In conclusion, we note that further substantive progress in the understanding of this hypothesis will only come from the study of nonspherical gravitational collapse, an area of study which has just begun [3,4].

This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada. It is a pleasure to thank Werner Israel for a number of enlightening discussions.

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[7] I have used PC MACSYMA 415.25 with the DOS Extender for an Intel 486. By expand, in the text, I mean the MACSYMA expand function. (The patch URICCI.MAC is flawed. For a correction contact the author.)

[8] To avoid singular points, $(2/c)(1-2m/r)^{-1}$ must be single valued, continuous, and satisfy the usual Lipshitz conditions. See, for example, T. V. Davies and E. M. James, *Nonlinear Differential Equations* (Addison-Wesley, New York, 1966).

[9] See T. Zannias, Phys. Rev. D **44**, 2397 (1991); D. M. Eardley, J. Isenberg, J. Marsden, and V. Moncrief, Commun. Math. Phys. **106**, 137 (1986).

[10] See K. Lake and T. Zannias, Phys. Rev. D **41**, 3866 (1990), for precise geometric criteria associated with the development of naked strong-curvature singularities in self-similar spacetimes.

[11] See, for example, L. Perko, *Differential Equations and Dynamical Systems* (Springer-Verlag, New York, 1991).

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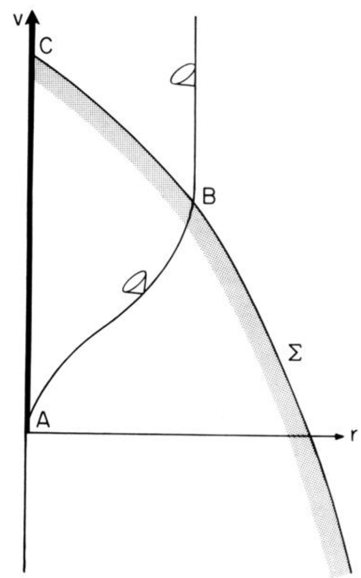
[13] See Ref. [2] for discussions of a number of special cases.

[14] A particularly useful reference is A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Qualitative Theory of Second-Order Dynamical Systems* (Wiley, New York, 1973). See, in particular, theorem 65.

[15] See, for example, R. M. Wald, *General Relativity* (Univ. Chicago Press, Chicago, 1984).

[16] There is no evidence that $m_2 > (1-2m_1)^2/16$ is generic, though this inequality holds for homogeneous collapse. In this Letter I have not included a discussion of nonradial null geodesics. These do not alter the conclusions in a substantive way.



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