PHYSICAL REVIEW LETTERS

VOLUME 68 | 25 MAY 1992 | NUMBER 21

Soliton Generation for Initial-Boundary-Value Problems

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(Received 11 December 1991; revised manuscript received 20 April 1992)

The solution of the initial-boundary-value problem of integrable nonlinear evolution equations, with the spatial variable on a half-infinite line, can be reduced to the solution of a linear integral equation. The asymptotic analysis of this equation for large t shows how the boundary conditions can generate solitons.

PACS numbers: 03.40.Kf

Most physical problems are naturally formulated as boundary-value problems. Such a physical problem arises, for example, in a certain laboratory study of water waves [1]; this situation can be modeled by the Korteweg–de Vries (KdV) equation on a semi-infinite line, with $q(x,0)=0$ and $q(0,t)$ given. This problem is well posed [2]; furthermore, numerical studies [3,4] indicate soliton generation, which is in agreement with the experimental observations. But an analytical confirmation of these numerical results remains open.

The soliton theory for solving the initial-value problem of integrable equations has been well developed; it reduces the solution of a given nonlinear equation to the formulation of a Riemann-Hilbert problem, which can be solved via a linear integral equation. The question of extending this theory for solving initial-boundary-value problems on a half-infinite line remained open for a rather long time (see, for example, [5]; the case of particular homogeneous boundary conditions was studied in [6–9]). One of the authors [10] recently presented a method for linearizing such problems. This method reduces the solution of the given nonlinear equation to the solution of two inverse problems, one associated with the x part of the underlying Lax pair, and the other with the t part of the Lax pair. Each of these problems can be formulated as a Riemann-Hilbert (RH) problem, and each of them can be solved through a linear integral equation. Before one can solve the x part, one must first obtain the so-called scattering data by solving the t part. Thus, while the solution of an initial-value problem reduces to the solution of one RH problem, the solution of an initial-boundary-value problem was reduced to the solution of two RH problems. This creates certain technical difficulties, in particular, for extracting information about the physically important question of the large-t behavior of the solution.

Here we first review the formulation of the above two RH problems. Then we show that it is possible to reduce them to a new *single* RH problem. It is remarkable that the solution of initial-boundary-value problems can also be obtained by solving only one RH problem (just like the case of initial-value problems).

This method is illustrated for the nonlinear Schrödinger (NLS) equation; it is also indicated how it can be applied to the KdV equation and to the N-wave interaction equations. It was shown in [11] that initial-boundary-value problems on the semi-infinite line are similar to certain forced problems, where the forcing is of a distribution type. We therefore expect that this method can also be used for the linearization of such forced problems.

Our scheme is a generalization of the method used for solving problems on the infinite line. We expect that a similar generalization of the method used for the periodic problem will linearize the problem on the finite domain.

Let us summarize the main steps of the method as applied to the NLS equation

$$iq_t+q_{xx}-2\lambda|q|^2q=0,\quad x,t\in[0,\infty);\ \lambda=\pm 1\,, \qquad (1)$$

where $q(x,0)$ and $q(0,t)$ are given, they decay for large x and t, and they satisfy the necessary compatibility condition to ensure the existence of solution at $x=0$, $t=0$.

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The cases $\lambda=1$ and $\lambda=-1$ will be referred to as the defocusing and focusing cases, respectively. The NLS equation is the compatibility condition [12] of the Lax pair [13]

$$w_x+ikJw=Qw\,,\quad w_t+Uw=0, \tag{2a}$$

$$U=2ik^2J+i\lambda|q|^2J-2kQ+iQ_xJ\,, \tag{2b}$$

where $J=\mathrm{diag}(1,-1)$ and Q is an off-diagonal matrix with 12 and 21 entries given by q and $\lambda\bar{q}$, respectively (throughout this paper the overbar denotes complex conjugate).

The first step of the method involves solving a direct and an inverse problem associated with the x part of the Lax pair, i.e., Eq. (2a). Let the 2×2 matrices $\varphi(x,t,k)$ and $\psi(x,t,k)$ be the solutions of Eq. (2a) specified by the boundary conditions $\varphi(0,t,k)=I$ and $\lim_{x\to\infty}\exp(ikxJ)\times\psi(x,t,k)=I$, respectively. Writing Volterra integral equations for these eigenfunctions, and letting $\Phi=\varphi\times\exp(ikxJ)$, $\Psi=\psi\exp(ikxJ)$, it follows that Φ and Ψ have analytic extensions in the complex k plane given by $\Phi=(\Phi^+,\Phi^-)$ and $\Psi=(\Psi^-,\Psi^+)$. This notation means that the first column of the matrix Φ is analytic for $k_I\geq0$ (where $k=k_R+ik_I$); similarly the second column of Φ is analytic for $k_I\leq0$. These matrices are related by the scattering equation $\Phi=\Psi\exp(-ikx\hat{J})[\psi(0,t,k)]^{-1}$, where $\hat{J}f$ denotes the commutator of J with f and hence $(\exp\hat{J})f=(\exp J)f\exp(-J)$. Rewriting this equation we obtain the RH problem

$$(\Psi^-(x,t,k),\Phi^-(x,t,k))=\left(\frac{\Phi^+(x,t,k)}{\psi_2^+(0,t,k)},\frac{\Psi^+(x,t,k)}{\psi_2^+(0,t,k)}\right)G(x,t,k),\quad k\in\mathbb{R}, \tag{3}$$

where the jump matrix G has entries $G_{11}=G_{22}=1$, $G_{12}=-\psi_1^+(0,t,k)\exp(-2ikx)$, and $G_{21}=\psi_2^-(0,t,k)\times\exp(2ikx)$. We shall denote this RH problem by $\chi^-=\chi^+G$ and we shall refer to it as the x-RH problem.

We note that there exist the symmetry relations

$$\overline{\Phi_1^+(k)}=\Phi_2^-(\bar{k}),\quad \overline{\Phi_2^+(k)}=\lambda\Phi_1^-(k)\,,$$

$$\overline{\Psi_1^-(k)}=\Psi_2^+(\bar{k}),\quad \overline{\Psi_2^-(k)}=\lambda\Psi_1^+(\bar{k})\,.$$

Because of the relation

$$\psi_2^-(0,t,k)=\overline{\psi_1^+(0,t,k)}\,,$$

the Riemann-Hilbert problem (3) is always solvable (i.e., there exists a vanishing lemma [14]). To solve this RH problem one needs $\psi_1^+(0,t,k)$ for $k\in\mathbb{R}$, the zeros of $\psi_2^+(0,t,k)$ for $k_I>0$, and the k derivative of ψ_2^+ at these zeros. Having these data, which are usually called the scattering data, Eq. (3) can be solved via a linear integral equation.

The second step of the method involves finding $\psi(0,t,k)$. The eigenfunction $\psi(x,t,k)$ satisfies $\psi_t+U\psi=\psi c(t)$. Evaluating this equation as $x\to\infty$, it follows that $c(t)=2ik^2J$. Hence $\psi_t(0,t,k)$ solves

$$\psi_t(0,t,k)+2ik^2[J,\psi(0,t,k)]=\hat{Q}(0,t,k)\psi(0,t,k)\,, \tag{4}$$

where

$$\hat{Q}(0,t,k)=2kQ(0,t)-i\lambda|q|^2(0,t)J-iQ_x(0,t)J\,.$$

The main idea of the method of [10] is to solve Eq. (4) by formulating for this equation an inverse scattering problem. Let $\hat{\Phi}(t,k)$ and $\hat{\Psi}(t,k)$ satisfy the same differential equation as $\psi(0,t,k)$ [i.e., Eq. (4)], and let them be specified by the boundary conditions $\hat{\Phi}(0,k)=I$ and $\lim_{t\to\infty}\hat{\Psi}(t,k)=I$, respectively. (We use the caret notation to help the reader appreciate the analogy between Φ,Ψ and $\hat{\Phi},\hat{\Psi}$.) Writing Volterra integral equations for these eigenfunctions, it follows that $\hat{\Phi}=(\hat{\Phi}^+,\hat{\Phi}^-)$ and $\hat{\Psi}=(\hat{\Psi}^-,\hat{\Psi}^+)$, where when + occurs with a caret it denotes analyticity in the first and third quadrants of the complex k plane (similarly when − occurs with a caret it denotes analyticity in the second and fourth quadrants). The matrices $\hat{\Phi}$ and $\hat{\Psi}$ are related by the scattering equation $\hat{\Phi}=\hat{\Psi}\exp(-2ik^2t\hat{J})[\hat{\Psi}(0,k)]^{-1}$. Rewriting this equation we obtain the RH problem

$$(\hat{\Psi}^-(t,k),\hat{\Phi}^-(t,k))=\left(\frac{\hat{\Phi}^+(t,k)}{\hat{\Psi}_2^+(0,k)},\frac{\hat{\Psi}^+(t,k)}{\hat{\Psi}_2^+(0,k)}\right)\hat{G}(t,k),\quad k^2\in\mathbb{R}, \tag{5}$$

where the jump matrix $\hat{G}$ has entries $\hat{G}_{11}=\hat{G}_{12}=1$, $\hat{G}_{12}=-\hat{\Psi}_1^+(0,k)\exp(-4ik^2t)$, and $\hat{G}_{21}=\hat{\Psi}_2^-(0,k)\times\exp(4ik^2t)$. We shall denote this RH problem by $\hat{\chi}^-=\hat{\chi}^+\hat{G}$ and we shall refer to it as the t-RH problem. Because of the symmetry

$$\overline{\hat{\Psi}_2^-(t,k)}=\lambda\hat{\Psi}_1^+(t,\bar{k})\,,$$

this RH problem is also always solvable.

The matrices $\psi(0,t,k)$, $\hat{\Psi}(t,k)$, and $\hat{\Phi}(t,k)$ satisfy the same ordinary differential equation (4); thus in the domains of their definitions they are simply related:

$$\psi^+(0,t,k)=\frac{\psi_2^+(0,0,k)}{\hat{\Psi}_2^+(0,k)}\hat{\Psi}^+(t,k),\quad k\in I, \tag{6a}$$

$$\psi^-(0,t,k)=\frac{\psi_1^-(0,0,k)}{\hat{\Psi}_1^-(0,k)}\hat{\Psi}^-(t,k),\quad k\in \mathrm{IV}\,, \tag{6b}$$

$$\psi^+(0,t,k)=\psi_2^+(0,0,k)\hat{\Phi}^-(t,k)+\psi_1^+(0,0,k)\hat{\Phi}^+(t,k)\times\exp(-4ik^2t),\quad k\in\mathrm{II}\,, \tag{7a}$$

$$\psi^-(0,t,k)=\psi_1^-(0,0,k)\hat{\Phi}^+(t,k)+\psi_2^-(0,0,k)\hat{\Phi}^-(t,k)\times\exp(4ik^2t),\quad k\in\mathrm{III}\,. \tag{7b}$$

We note that Eqs. (6b) and (7b) are consistent with the underlying symmetry conditions. Using the relations (6) and (7) it is possible to rewrite the t-RH problem in a more convenient form. To achieve this we first introduce some notation. Let

$$s_1^-(k)=\psi_1^-(0,0,k),\quad s_2^-(k)=\psi_2^-(0,0,k),$$
$$s_1^+(k)=\psi_1^+(0,0,k),\quad s_2^+(k)=\psi_2^+(0,0,k). \tag{8}$$

We note that the s's can be evaluated in terms of the initial data $q(x,0)$. It turns out that the t-RH problem involves the following quantities:

$$c(k)=\frac{r}{(s_2^+)^2(1-rs_1^+/s_2^+)},\quad b(k)=\frac{s_1^+}{s_2^+},$$
$$r(k)=\frac{\hat{\Psi}_2^-(0,k)}{\hat{\Psi}_1^-(0,k)}. \tag{9}$$

Using the definitions (8) and (9) the t-RH problem can be written as

$$\hat{Y}^-(t,k)=\hat{Y}^+(t,k)F(t,k),\quad k^2\in\mathbb{R}, \tag{10}$$
$$\hat{Y}\to I \text{ as } k\to\infty,$$

where $\hat{Y}$ is given by

$$\left[\frac{\hat{\Phi}^+(t,k)}{s_2^+(k)},\psi^+(0,t,k)\right],\quad \left[\frac{\hat{\Psi}^-(t,k)}{\rho(k)},\psi^+(0,t,k)\right],$$
$$\left[\psi^-(0,t,k),\frac{\hat{\Psi}^+(t,k)}{\nu(k)}\right],\quad \left[\psi^-(0,t,k),\frac{\hat{\Phi}^-(t,k)}{s_1^-(k)}\right],$$

for k in I,...,IV, respectively, where $\rho(k)=\hat{\Psi}_1^-(0,k)s_2^+(k)-\hat{\Psi}_2^-(0,k)s_1^+(k)$, $\nu(k)=\overline{\rho(\bar{k})}$. The jump matrix F has the following form: For $k\in i\mathbb{R}^+$, $F_{11}=F_{22}=1$, $F_{12}=0$, $F_{21}=c(k)\exp(4ik^2t)$; for $k\in i\mathbb{R}^-$, $F_{11}=F_{22}=1$, $F_{21}=0$, $F_{12}=-\lambda\overline{c(\bar{k})}\exp(-4ik^2t)$; for $k\in\mathbb{R}^+$, $F_{11}=1$, $F_{22}=1/|s_2^+|^2$, $F_{12}=-b(k)\exp(-4ik^2t)$, $F_{21}=\lambda\overline{b(k)}\exp(4ik^2t)$; for $k\in\mathbb{R}^-$, $F_{22}=1$, $F_{11}=1+F_{12}F_{21}$, $F_{12}=[b(k)-\lambda\overline{c(k)}]\exp(-4ik^2t)$, $F_{21}=[c(k)-\lambda\overline{b(k)}]\exp(4ik^2t)$.

The RH problem (10) is uniquely defined in terms of the initial data $q(x,0)$ (which specify the s's) and of $r(k)$, which is in principle specified in terms of the boundary data $q(0,t)$. $r(k)$ is analytic for $k\in$ II$\cup$IV; also because $r=s_2^-/s_1^-$ for $k\in$ IV, $r(k)$ has analytic continuation for $k\in$ III.

The final step of the method involves formulating a new RH problem, whose solution gives the solution of both the x- and t-RH problems: Let $Y(x,t,k)$ satisfy the same jump conditions as $\hat{Y}(t,k)$ [Eq. (10)], where $\exp(\pm 4ik^2t)$ is replaced by $\exp(\pm 4ik^2t\pm 2ikx)$. Then

$$\Psi^+(x,t,k)=Y(x,t,k)(0,1)^T,\quad k\in \text{I},$$
$$q(x,t)=2i\lim_{k\to\infty}[kY(x,t,k)]_{12},\quad k\in\text{I}. \tag{11}$$

In the focusing case ($\lambda=-1$), Y may have poles and one needs some additional information about the position of these poles: Let $k_j\in$ II, $1\le j\le N$, satisfy $\hat{\Psi}_1^-(0,k_j)=0$; then Y has poles at k_j and at $\bar{k}_j$. We define $\tilde{Y}$ by

$$\tilde{Y}=Y\operatorname{diag}\left[\prod_{j=1}^{N}(k-k_j),\prod_{j=1}^{N}(k-\bar{k}_j)\right].$$

Then $\tilde{Y}$ has the following properties: (a) does not have poles; (b) satisfies a RH problem with the same jump conditions as that of Y where r is replaced by $\tilde{r}=r\prod(k-k_j)/\prod(k-\bar{k}_j)$; (c) $\tilde{Y}\sim k^NI+O(k^{N-1})$ as $k\to\infty$; and (d) $\tilde{Y}$ satisfies

$$\tilde{Y}(x,t,k_j)(1,-c_j\exp[4itk_j^2+2ixk_j])^T=0,$$
$$c_j=\frac{\hat{\Psi}_2^-(0,k_j)}{\hat{\Psi}_{1,k}^-(0,k_j)}\frac{\prod_{i=1,i\neq j}^{N}(k_j-k_i)}{\prod_{i=1}^{N}(k_j-\bar{k}_i)}. \tag{12}$$

To solve the RH problem for $\tilde{Y}$, one first solves a canonical RH problem for X, which satisfies only conditions (a) and (b) above, as well as $X\to I$ as $k\to\infty$. Having obtained X, it is straightforward to find $\tilde{Y}$.

We now discuss the long-time behavior of the above formalism. For simplicity we assume $q(x,0)=0$. It is possible to show that as $t\to\infty$

$$X(x,t,k)=[I+O(1/\sqrt{t})]\operatorname{diag}(a(k),a^{-1}(k)),$$
$$a(k)=\exp\left[\frac{1}{2\pi i}\int_{-\infty}^{k_0}\frac{\ln[1+|r(k')|^2]}{k-k'}dk'\right], \tag{13}$$

where $k_0=-x/4t$. Using this estimate it follows that in the defocusing case $q(x,t)$ disperses away like $1/\sqrt{t}$, as $t\to\infty$. In the focusing case, the main contribution comes from the discrete spectrum. In particular if $N=1$, then

$$q(x,t)=-\frac{2\eta\exp[-2i\xi x-4i(\xi^2-\eta^2)t-i\varphi]}{ch(\eta x+4\xi\eta t-\Delta)}+O(1/\sqrt{t}),$$

where

$$\varphi=-\frac{\pi}{2}+\arg\tau+\frac{1}{\pi}\int_{-\infty}^{k_0}\frac{\ln[1+|r(k)|^2]}{(k-\xi)^2+\eta^2}(k-\xi)dk,$$
$$\tau=\operatorname{res}_{k_1}r(k)$$
$$\Delta=-\ln 2\eta+\ln|\tau|-\frac{\eta}{\pi}\int_{-\infty}^{k_0}\frac{\ln[1+|r(k)|^2]}{(k-\xi)^2+\eta^2}dk,$$
$$k_1=\xi+i\eta.$$

We note that since $\eta>0$ and $\xi<0$, the solitons move away from the boundary. Also, in contrast to the solution to the Cauchy problem for $x\in(-\infty,\infty)$, here τ is not arbitrary but depends on $r(k)$.

We conclude with some remarks.

(1) It is straightforward to make the above exact

analysis rigorous: The direct problem satisfies Volterra integral equations; thus for $q(x,0)$ and $q(0,t)$ in $L^1(\mathbb{R}^+)$, these equations are always solvable. Also, both the x and t inverse problems are formulated via RH problems which satisfy a certain Schwarz reflection symmetry; therefore if the jump functions are in $H^1(\mathbb{R})$ these RH problems are always solvable [14]. To make the above asymptotic analysis rigorous [15] one should use the recent work of [16].

(2) There exist conceptual similarities between the problem studied here and the generation of dromions for the Davey-Stewartson I equation [17]: In both problems, it is the spectrum of the t part of the Lax pair that determines the emergence of coherent structures.

(3) It follows from the above discussion that the fundamental RH problem associated with the solution of an initial-boundary-value problem is defined on a contour specified by the t part of the Lax pair. In the case of the NLS this part contains K^2; thus this RH problem is defined on $k^2 \in \mathbb{R}$. In the case of the N-wave interactions the evolution of the scattering data T satisfy $T_t = ik\hat{C}T - T\hat{C}\hat{J}^{-1}Q(0,t)$, $Q(x,t)$ is an $N\times N$ off-diagonal matrix, and J,C are $N\times N$ diagonal matrices with real, distinct entries. Thus, for the N-wave interactions the fundamental RH problem is defined on $k\in\mathbb{R}$. In the case of the Korteweg-de Vries equation the scattering data satisfy $T_t - 4ik^3\hat{J}T = \hat{Q}(k,t)T$, where $T(t,k)$ is a 2×2 matrix, $J = \text{diag}(1,-1)$, and $\hat{Q}(k,t)$ is a rational function of k which depends on $q(0,t)$, $q_x(0,t)$, and $q_{xx}(0,t)$; thus the RH problem is now defined on $k^3 \in \mathbb{R}$.

This is part of a large program of study with several colleagues, including J. Bona, S. Brenner, L. Sung, and S. Venakides. The authors are grateful to I. M. Gel'fand for his constant encouragement. This work was partially supported by the National Science Foundation under Grant No. DMS-8803471 and by the Air Force Office of Scientific Research under Grant No. 87-0310.

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