Empirical Determination of Universal Multifractal Exponents in Turbulent Velocity Fields

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(Received 14 June 1991)

It is now apparent that the two principal models of turbulence (the "beta" and "lognormal" models) are the extremes of a continuous family of (stable, attractive, hence "universal") multifractals characterized by Levy indices $\alpha=0$ and 2, respectively. Using a technique called double trace moment analysis, and turbulent velocity data, we empirically obtain $\alpha \approx 1.3 \pm 0.1$: As has long been suspected, turbulence really is "in between" the β and lognormal models. This describes the entire hierarchy of singularities of the Navier-Stokes equations.

PACS numbers: 47.25.−c

In three-dimensional hydrodynamic turbulence, cascade processes have long been believed to provide the primary mechanism of injection of energy flux from large to small scales. Since the 1960's specific models [1–3] of cascades have been developed. They model the nonlinear dynamical processes by concentrating various conserved fluxes (e.g., the energy flux) from large to smaller and smaller volumes, leading to highly intermittent fields. Such models are (deceptively) simple; they generically give rise to singular multiscaling-multifractal measures.

In spite of the generality of this theoretical result, not enough effort has been made to empirically determine the infinite hierarchy of dimensions (we will compare our results to Ref. [3]). With the recognition that cascade processes [4] have stable and attractive universality classes [5,6] specified by only two parameters (three for nonconservative processes), the problem can be attacked with more powerful tools: Only a few rather than an infinite number of parameters need to be determined. The most basic of these [7] (α) interpolates between the two oldest and best studied cascade models, the "β model" [1] ($\alpha=0$), and the "lognormal model" [8,9] ($\alpha=2$). Indeed, we shall see that the empirical value found below ($\alpha \approx 1.3$) corresponds exactly to the common observation that turbulence is "in between" (e.g., Refs. [10] and [11]). Here we apply the first technique, double trace moments (DTM) [12], specifically designed to exploit this universality, yielding robust parameter estimates; we apply it to turbulent wind fields.

When a cascade has proceeded over a scale ratio $\lambda = L/l$ (the ratio of the largest scale of interest to the smallest scale) the density of the conserved energy flux [13] (ε) has the singular behavior [5,14,15]

$$\varepsilon_\lambda \approx \lambda^\gamma, \quad \lambda > 1\,. \tag{1}$$

Where $\lambda \to \infty$ (or $l \to 0$), $\gamma > 0$ is the order of the singularity, and $\gamma < 0$ is the "order of the regularity." The probability distribution of singularities whose order exceeds γ and the related statistical moments will have the following scaling behavior [5]:

$$\Pr(\varepsilon_\lambda \geq \lambda^\gamma) \approx \lambda^{-c(\gamma)} \Longleftrightarrow \langle \varepsilon_\lambda^q \rangle \approx \lambda^{K(q)}\,, \tag{2}$$

where $c(\gamma)$ is the codimension function of the singularities ($\langle\,\rangle$ indicates ensemble averaging). It is related by a Legendre transform [14] to the scaling exponent [16] $K(q)$ associated with statistical moments. The function $c(\gamma)$ is a codimension since the probability measures the fraction of the (infinite-dimensional) probability space occupied by the singularities exceeding the order γ. When individual realizations are studied on spaces with dimension D (i.e., D-dimensional cuts of the infinite-dimensional stochastic process), and when $c(\gamma) \leq D$, this statistical exponent has a simple geometrical [14,17] interpretation: It is the scaling exponent of the fraction of the D volume occupied by singular values greater than γ. The relations between the turbulence notation [18] used here and the strange attractor $f_D(\alpha_D)$ and $\tau_D(q)$ notation [15] (the subscript D explicitly emphasizes the dependence of α, f, and τ on the dimension of the observing space D) are $f_D(\alpha_D) = D - c(\gamma)$ and $\tau_D(q) = K(q) - (q-1)D$, with $\alpha_D = D - \gamma$. The turbulence notation used is necessary when dealing with stochastic processes because γ, c, and K are intrinsic, contrary to α_D, f_D, and τ_D which diverge with $D \to \infty$; it also avoids introducing negative ("latent") dimensions [19] when $c(\gamma) > D$.

The basic idea of the DTM technique is to generalize the application of statistical methods to the quantity $\varepsilon_{\lambda'}^\eta$. This is done by taking the ηth power of $\varepsilon_{\lambda'}$ at the scale ratio λ' (the outer or largest scale of interest to the smallest scale of homogeneity), and then studying the scaling behavior of the various qth moments at decreasing values of the scale ratio $\lambda \leq \lambda'$. The q,η double trace moment at resolution λ' and λ has the following multiple scaling behavior:

$$\begin{aligned} \Pi_{\lambda,\lambda'}^{(\eta)}(B_\lambda) &= \int_{B_\lambda} \varepsilon_{\lambda'}^\eta d^D x\,, \\ \mathrm{Tr}_\lambda(\varepsilon_{\lambda'}^\eta)^q &= \left\langle \sum_A (\Pi_{\lambda,\lambda'}^{(\eta)})^q \right\rangle \propto \lambda^{K(q,\eta)-(q-1)D}\,. \end{aligned} \tag{3}$$

The sum over A is done at resolution λ, i.e., on a (more or less optimal) covering of the (D-dimensional) balls B_λ's. The integration over B_λ rescales the fields and corresponds then to the "dressed" quantities discussed in Refs. [5], [12], and [18]. When $\eta=1$ the right-hand side of Eq. (3) reduces to the usual trace moments [5] (which are themselves simply ensemble averages of the usual partition function).

The scaling exponent $K(q,\eta)$ is related to $K(q,1) \equiv K(q)$ by

$$K(q,\eta)=K(q\eta,1)-qK(\eta,1)\,. \tag{4}$$

Using the universality classes for $K(q,1)$ gives the expression for $K(q,\eta)$:

$$K(q,\eta)=\eta^\alpha K(q,1)=\begin{cases}\dfrac{C_1}{\alpha-1}\eta^\alpha(q^\alpha-q), & \alpha\neq 1,\\ C_1\eta q\ln(q), & \alpha=1,\end{cases} \tag{5}$$

with $0\leq\alpha\leq 2$, and $q>0$ for $\alpha\neq 2$. By keeping q fixed (but different from the special values 0 or 1) the slope of $|K(q,\eta)|$ as a function of η on a log-log graph gives the value of the parameter α, which with the help of the intercept yields C_1. Varying q then allows for a systematic verification of Eq. (5), and hence the universality hypothesis.

For sufficiently large q, Eq. (5) breaks down. Two critical values must be considered. The first is the maximum moment q_s that can be estimated by a finite number N of samples. Defining the "sampling dimension" [6,20] $D_s = \log N/\log\lambda$, we have $q_s=[(D+D_s)/C_1]^{1/\alpha}$. The second critical value q_D [the solution of $K(q_D,1)=(q_D-1)D$] arises when multifractals are averaged over sets of dimension D with scales much larger than the inner scale of homogeneity (here, the dissipation scale). For $q\geq q_D$ the empirical moments will diverge and empirical estimates will be subject to spurious (or "pseudo") scaling [5]. Whenever $\max(q,q\eta)\geq\min(q_s,q_D)$, $K(q,1)$ becomes a linear function of q and Eq. (4) indicates then that $K(q,\eta)$ becomes independent of η. Reference [12] contains considerable theoretical and numerical investigation of the DTM technique, as well as error estimates. An important advantage of the DTM over other analysis techniques is that the determination of α is invariant under the general transformation $\gamma\rightarrow a\gamma+b$, e.g., to arbitrary powers of non-normalized processes.

We now discuss the application of the DTM method to turbulent velocity measurements made by Gagne at the ONERA wind tunnel, Modane, with a high-resolution hot wire anemometer sampling at 10 kHz. We analyzed $N=190$ samples each of scale ratio $\approx 2^{12}$ in the inertial (scaling) regime (hence $D_s=0.63$), and $\approx 2^2$ in the dissipation region; this corresponds to the smallest scale ≈ 8 mm. The energy spectrum of the time series is shown in Fig. 1. The velocity amplitude signal is then passed through a filter that weights its Fourier components by $\omega^{1/3}$ (ω is a frequency). This removes the $\lambda^{-1/3}$ Kolmogorov scaling yielding the conservative quantity $\varepsilon^{1/3}$ related to the velocity by [21]

$$\Delta v_\lambda\approx\varepsilon_\lambda^{1/3}\lambda^{-1/3}\,. \tag{6}$$

Figures 2 and 3 show the results when the DTM technique is applied to various values of q,η. As long as η and $q\eta$ are below $q_s\approx 4$–5, the plots of $\log|K(q,\eta)|$ vs

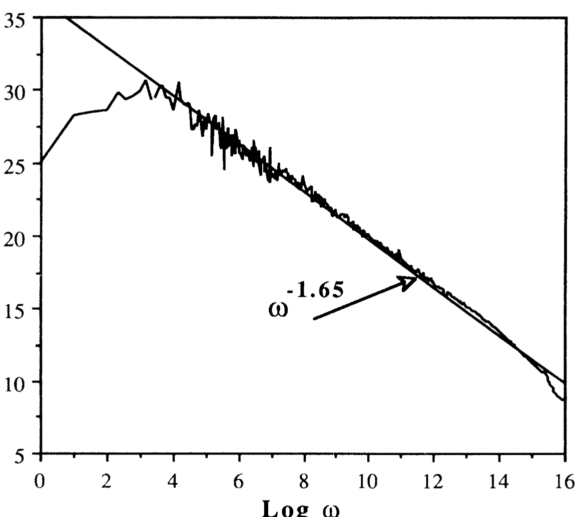

FIG. 1. The velocity spectrum averaged over 190 samples each of length 2^{14} times the finest resolution (which corresponds to 10 kHz). The spectrum $E(\omega)\approx\omega^{-\beta}$ (ω is the frequency) with $\beta\approx 1.65$ is shown for comparison.

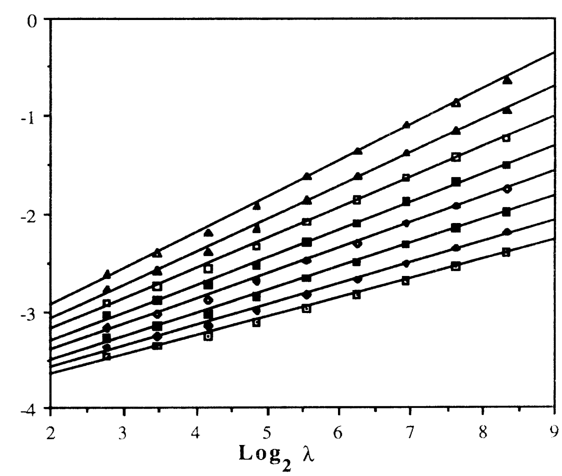

FIG. 2. Log of the double trace moment as a function of $\log\lambda$ for various values of q,η, showing that the scaling is well respected. The extreme four octaves which are part of the dissipation range were not used.

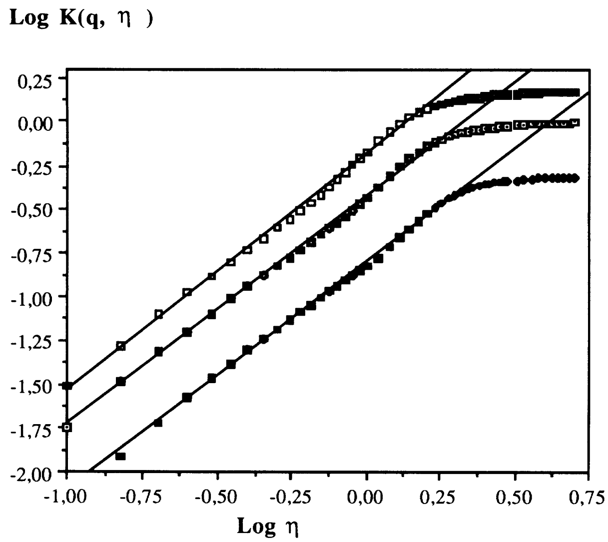

FIG. 3. Curves of $\log_{10}|K(q,\eta)|$ vs $\log_{10}\eta$, for $q=2.5$, 2, and 1.5 (top to bottom). The curves are nearly parallel and for $\eta q < q_s \approx 5$ have slopes $\alpha \approx 1.3 \pm 0.1$. Furthermore, the values of $\log_{10}|K(h,1)|$ give $C_{1\varepsilon} \approx 0.25 \pm 0.05$.

$\log\eta$ (Fig. 3) are very straight, as expected for universal multifractals, with slopes and intercepts yielding $\alpha \approx 1.3 \pm 0.1$ and $C_1 \approx 0.25 \pm 0.05$. For comparison with other empirical results [22], we may calculate the standard intermittency parameter μ which is the autocorrelation exponent for ε: $\mu = K(2,1)$. For the lognormal model ($\alpha=2$), we have $\mu = 2C_1$, whereas for the β model ($\alpha=0$), $\mu = C_1$. Here, with $\alpha=1.3$, we obtain $\mu \approx 1.55C_1 = 0.35 \pm 0.1$, which is exactly in the middle of the accepted range [10] 0.2–0.5 [11]. Finally, we may calculate $q_s \approx 4.2$, which is in excellent agreement with the breakdown of Eq. (5) for large q found in Fig. 3.

Because $\alpha \geq 1$, turbulence is an unconditionally hard multifractal process, i.e., high enough order statistical moments will diverge when energy fluxes are averaged over spaces with arbitrary dimensions (as long as the average is over scales much larger than the dissipation scale). This is in accord with empirical evidence from atmospheric data [23].

Leray [24], Von Neuman [25], and many others (e.g., Refs. [5], [14], and [26]) have pointed out the importance of characterizing the singularities of the Navier-Stokes equations. This experimental confirmation of universal multifractals in turbulence has fundamental implications for the high-Reynolds-number limit, since with the help of three fundamental exponents it characterizes the possible class of solutions, in particular the entire hierarchy of singularities.

We acknowledge A. Davis, C. Hooge, P. Ladoy, J. P. Kahane, K. Pflug, G. Sarma, Y. Tessier, R. Viswanathan, B. Watson, and J. Wilson. We are grateful to Y. Gagne and E. Hopfinger, U. Frisch, and the DRET for providing the wind velocity data, and the Atmospheric Radiation Measurement Program Contract No. DE-FG03-90ER-61062 for partial financial support.

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