Cyclotron Emission from Nonuniformly Magnetized Plasmas

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A new quantitative representation of the generalized Kirchhoff's law relating the emission from each propagating branch to the absorption along the corresponding branch is established, including for the first time the effects of inhomogeneous magnetic fields on cyclotron and synchrotron radiation from mode conversion theory. The concept of optical depth is revised to include effects of reflection and conversion in addition to transmission. Via the use of a variational principle, the source distribution function for the inhomogeneous emitting layer is calculated.

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Cyclotron emission from both relativistic and nonrelativistic plasmas, correct estimates of which are critically important for power balance in fusion plasmas and for plasma diagnostics, has been discussed by many authors [1-6]. Nearly all have considered either a uniform plasma layer of finite extent with *ad hoc* assumptions about the effective thickness of the layer or a plasma with slowly varying plasma parameters. Prior to the analysis of the effects of mode conversion on such emission problems [6], however, the effects of the varying magnetic field have never been included consistently. It is the purpose of this paper to demonstrate the significant discrepancies between the generalized Kirchhoff's law (GKL) and conventional theory and to present for the first time the localized source distribution function.

Since the thickness of the cyclotron emission layer is nearly always determined by the variation of the magnetic field, the GKL supersedes virtually all previous cyclotron and synchrotron emission results. In the limit of completely absorbing layers, where blackbody conditions apply, the results coincide, but even in cases where the transmission coefficient is virtually zero, there may be substantial differences from classical results. In addition, radiation from a volume of plasma remote from the sampled volume may add to the directly emitted radiation through the mode conversion process.

The basic mode conversion tunneling equation with absorption may be written as [7]

$$\psi^{\mathrm{iv}}+\lambda^2 z\psi''+(\lambda^2 z+\gamma)\psi=h(z)(\psi''+\psi)\,, \tag{1}$$

where $-\infty<z<\infty$, $\lambda^2>0$. For the ion cyclotron harmonic, $\gamma>-1$, $h(z)=\frac{1}{2}\lambda^2\kappa Z'(\zeta)/Z(\zeta)$. $\gamma<-1$ for both the X mode at $\omega=2\omega_{ce}$ and the O mode at $\omega=\omega_{ce}$ and $h(z)\propto\zeta+\frac{7}{2}-1/F_{7/2}(\zeta)$ for the weakly relativistic cases where ζ and z are proportional to the distance from the resonance point and $Z(\zeta)$ and $F_q(\zeta)$ are the nonrelativistic and relativistic plasma dispersion functions. The dimensionless parameters λ, γ, and κ are functions of the plasma parameters n_e, B_0, R_0, T_i or T_e, and k_z which are given along with the scattering parameters both without (Ref. [7]) and with (Ref. [8]) absorption.

The solutions of Eq. (1) are denoted by ψ_k, and their adjoint solutions by $\Psi_k=\psi_k''+\psi_k$. The subscript denotes a solution which represents (1) a fast wave incident from the high magnetic field side, (2) a fast wave from the opposite side, (3) an incident slow (Bernstein) wave, or (4) an exponentially growing solution. We also define the constants $\eta=|\pi(1+\gamma)/2\lambda^2|$ and $\varepsilon=1-e^{-2\eta}$.

For the emission of radiation from an absorbing plasma, the emission equation is obtained from Eq. (1) by adding a source term, $s(z)$, to the right-hand side. The solution of this equation has only outgoing waves. The source distribution function is a localized function which is presumed to satisfy some normalization condition

$$\int_{-\infty}^{\infty}\rho(z)|s(z)|^2dz=P_0>0\,, \tag{2}$$

where $\rho(z)$ is a positive weight function.

The fraction of incident energy absorbed on each branch is given through the scattering parameters by [6,9]

$$A_k=1-|T_k|^2-|R_k|^2-|C_{k3}|^2,\quad k=1,2\,,$$

$$A_3=1-|R_3|^2-|C_{31}|^2-|C_{32}|^2\,,$$

where $|T_1|=|T_2|\equiv T$, and $C_{k3}=C_{3k}$ from reciprocity [9]. Using these expressions, the generalized Kirchhoff's law has been established to be [6]

$$\frac{E_1}{A_1}=\frac{E_2}{A_2}=\frac{E_3}{A_3}=I\,, \tag{3}$$

where E_k is the power emitted on branch k.

In the general proof of the GKL, Eq. (3), which was based on thermodynamics arguments, the incident radiation I from the walls of the chamber was in thermal equilibrium with the ambient plasma. This implies, however, that the walls and plasma are at the same temperature, so that the walls (which were presumed to be perfectly absorbing) radiate as a blackbody at that temperature, so that $I=I_{\mathrm{BB}}$ where I_{BB} is the radiated power from a blackbody [10]. The GKL is then written as

$$E_k=A_kI_{\mathrm{BB}},\quad k=1,2,3\,. \tag{4}$$

This global proof can now be extended to a local proof by moving the walls to infinity so that there are only outgoing waves, in which case the emission must be unchanged provided there is some (nonradiative) energy source to maintain the temperature.

The GKL of Eq. (4) is to be compared with the classical expression based on opacity arguments [2] such that

$$E=(1-e^{-\tau})I_{\mathrm{BB}}\,, \tag{5}$$

where the opacity τ is related to the transmission coefficient $T^2 = e^{-\tau}$ and given by

$$\tau = 2\int \mathrm{Im}[k(z)]dz , \qquad (6)$$

where $k(z)$ is the slowly varying wave number from the dispersion relation and the integral is across the emitting-absorbing layer. τ is generally attributed to absorption, but often erroneously so, since there is always a tunneling component in $\mathrm{Im}[k(z)]$ from mode conversion which is dominant in weak absorption but insignificant in strong absorption. The transmission coefficient in a mode conversion layer is $T^2 = e^{-2\eta}$, and independent of absorption. Numerical studies show that $\tau = 2\eta$ even in cases where $\mathrm{Im}(k)$ is not small and the validity conditions for WKB theory fail. We note that in mode conversion layers, τ may vary widely *even when there is no absorption.*

The kinds of discrepancies which occur in estimating emission from the classical formula are illustrated in Table I for ions and in Table II for electrons which are obtained from numerical solutions of Eq. (1). The first two rows list the ratios of the classical emission from Eq. (5) to the direct emission from Eq. (4) for several values of k_z (ion case) and T_e (electron case). This ratio is

$$r_k = (1 - e^{-\tau})/A_k . \qquad (7)$$

The plasma parameters for both cases were chosen to illustrate the transition from weak to strong absorption. The gradient scale length for the electron case is more representative of a torsatron than a tokamak.

The effects of the Bernstein wave on the emission are several, namely: (1) The radiation along the two fast wave branches in opposite directions is asymmetric [6,11]; (2) the radiation detected on either fast wave branch is actually the sum of the radiation directly along that branch from the cyclotron layer *and the radiation from a remote region* along the Bernstein branch which is partially mode converted into a fast wave; and (3) there is a substantial amount of radiation along the Bernstein branch which is traditionally ignored in transport estimates.

In the first case, the classical radiation expressions used in electron cyclotron emission (ECE) diagnostics would yield different temperatures from the two sides unless corrected for the different absorbed fractions. Examples of the asymmetry are given in the last row of Table II. If the layer is thin and reflection from a far wall is included, the source for the reflected wave will have a different strength from the direct source.

TABLE I. Ratio of direct ion cyclotron emitted power r_k from Eq. (7) and indirect ratio r'_k from Eq. (10) with $I'_{BB}/I_{BB} = 0.5$ vs k_z (in m^{-1}) with $n_e = 1.0\times10^{20}/\mathrm{m}^3$, $B_0 = 5$ T, $R_0 = 2$ m, and $T_i = 2$ keV.

	k_z					
	0	2	4	6	8	10
r_1	∞	71.5	15.7	5.97	3.57	2.335
r_2	∞	4.94	1.68	1.16	1.032	1.005
r'_1	2.00	1.97	1.88	1.71	1.56	1.40
r'_2	3.39	2.75	1.49	1.13	1.027	1.005

TABLE II. Ratio of direct electron synchrotron emitted power r_k from Eq. (7) and asymmetry ratio E'_2/E'_1 vs T_e (in eV) with $n_e = 4.0\times10^{19}/\mathrm{m}^3$, $B_0 = 3$ T, and $R_0 = 0.15$ m.

	T_e				
	100	200	300	400	500
r_1	3.15	1.95	1.53	1.32	1.20
r_2	2.02	1.36	1.17	1.10	1.06
E'_2/E'_1	0.933	0.909	0.921	0.943	0.962

Second, emission from the cyclotron layer and from a neighboring region in the plasma is inextricably mixed, further complicating the interpretation of ECE diagnostics. Ray tracing studies indicate that the emitted electron Bernstein wave is typically reabsorbed a few centimeters away from the source, so we may assume that a blackbody source at the same temperature radiates back along the ray path, and some is mode converted to the fast wave branches, adding to the direct emission. This total emission E'_k is given by

$$E'_k = (1 - T^2 - |R_k|^2 - |C_{k3}|^2)I_{BB} + |C_{3k}|^2 I'_{BB} , \qquad (8)$$

for $k = 1,2$, and where I'_{BB} is the blackbody emission from the Bernstein wave source point. If we assume $I_{BB} = I'_{BB}$ for electrons because of the proximity of the two sources, then the converted terms cancel, so that

$$E'_1 = (1 - T^2)I_{BB}, \quad E'_2 = (1 - T^2 - |R_2|^2)I_{BB} , \qquad (9)$$

since $R_1 = 0$ [8]. For branch 1, then, the classical result is reestablished, but for different reasons and only for the case where $I'_{BB} = I_{BB}$, while there remains a discrepancy for branch 2. In Table II, we give the ratio of the emission on branch 2 to the classical result, which is also the measure of the asymmetry. As T_e approaches zero, the asymmetry vanishes as the tunneling layer becomes transparent and $T \rightarrow 1$, while as T_e becomes large, the asymmetry vanishes because the plasma becomes a blackbody radiator, so the maximum asymmetry occurs at some intermediate temperature. For the ions, we do not expect $I_{BB} = I'_{BB}$ since the ion Bernstein wave will either be Landau damped or propagate to the next lower harmonic, so that the temperature at the remote point will typically differ by a significant amount. To show the kinds of effects this would introduce we show in the last two rows of Table I the ratio of the classical emission to the sum of the direct and indirect emission for a case where we have chosen $I'_{BB} = \frac{1}{2} I_{BB}$ for illustration, where, by analogy to Eq. (7), we write

$$r'_k = (1 - e^{-\tau})I_{BB}/E'_k . \qquad (10)$$

The enhancement is important for weak to moderate absorption, but vanishes in the limit of strong absorption.

In the final case, the Bernstein wave radiation in a fusion plasma represents a source of nonclassical energy transport, since the energy transport due to this branch is greater than or equal to the X-mode radiated power, but the Bernstein wave will not exit the plasma. For electrons, the energy is transported a few centimeters as the Bernstein wave returns to the same harmonic, but displaced vertically, while for ions the distance will be larger. Either process would appear as anomalous transport since it is unrelated to collisional processes, and may be a significant source of anomalous diffusion since the propagation distance for electron Bernstein waves is hundreds of Larmor radii, and thus large compared to the collisional step size.

Whereas direct calculations of emission from a source model in both homogeneous and weakly inhomogeneous media have been previously executed, there are no previous theories of the source distribution function from a mode conversion layer where the coupling between the fast and slow waves must be taken into account. Following the spirit of the fluctuation-dissipation theorem or Kirchhoff's law, which implies that every sink is a source and vice versa, we obtain the source distribution from a known sink distribution. In this analysis, we first need to find integral expressions for the absorption and emission in terms of sink and source distributions.

An energy flux conservation law for the solutions, f_k, without absorption may be obtained from the expression [7]

$$P[f]=f'''(f''+f)^*-f''f^{*\prime}-\gamma f^*f'-\text{c.c.}=\text{const}\,,$$

so that $dP/dz=0$. For the corresponding equation with absorption, Eq. (1), one can show that

$$\frac{dP[\psi_k]}{dz}=(h-h^*)|\Psi_k(z)|^2,\quad k=1,2,3\,. \tag{11}$$

On the other hand, by the substitution of the asymptotic forms for ψ_k in the expression for P, one can show that

$$P[\psi_k(\infty)]-P[\psi_k(-\infty)]=2\pi i\lambda^2\varepsilon(A_k/a_k)\,, \tag{12}$$

which, along with Eq. (11), gives

$$A_k=a_k\int_{-\infty}^{\infty}w(z)|\Psi_k|^2dz\,, \tag{13}$$

where $a_1=e^{-2\eta}$, $a_2=1$, $a_3=\varepsilon e^{-2\eta}$, and $w(z)\equiv\text{Im}[h(z)]/\pi\lambda^2\varepsilon>0$ is the localized absorption function.

From the Green-function representation [9] of the solution of Eq. (1) with a source, $s(z)$, the asymptotic behavior of the radiation field is determined by the integrals

$$S_k=\int_{-\infty}^{\infty}B_k(y)s(y)dy\,, \tag{14}$$

where the functions $B_k(y)$ are related to $\Psi_k(y)$ through some linear transformations [9], and by the asymptotic behavior of the ψ_k. From the fact that the radiation field has only outgoing waves, the asymptotic solutions lead eventually to the result that

$$E_k=a_k\left|\int_{-\infty}^{\infty}s(z)\Psi_k(z)dz\right|^2. \tag{15}$$

Since E_k is a functional of $s(z)$ and A_k is independent of $s(z)$, and since I_{BB} is the maximum possible radiation which maximizes the right-hand side of Eq. (4), we must choose $s(z)$ to maximize the left-hand side. Considering first the emission on a single branch k, we have

$$\left|\int_{-\infty}^{\infty}s(z)\Psi_k(z)dz\right|^2=\max\,. \tag{16}$$

The natural constraint on $s(z)$, so that the integral is bounded, is given by Eq. (2). If we introduce the scalar product of two integrable functions as

$$\langle f|g\rangle=\int_{-\infty}^{\infty}w(z)f^*(z)g(z)dz\,, \tag{17}$$

then applying the Cauchy-Schwarz inequality to the integral of Eq. (16) we have

$$\left|\int_{-\infty}^{\infty}s(z)\Psi_k(z)dz\right|^2\le\langle\Psi_k|\Psi_k\rangle\int_{-\infty}^{\infty}\frac{|s(z)|^2}{w(z)}dz\,.$$

This inequality reduces to equality only when f and g are linearly dependent, so the extremum in Eq. (16) occurs when $s(z)\propto w(z)\Psi_k^*$, and the bound on $s(z)$ is given by

$$\int_{-\infty}^{\infty}\frac{|s(z)|^2}{w(z)}dz=P_0\,, \tag{18}$$

so that we can see that $\rho(z)=1/w(z)$ in Eq. (2).

Following this prescription, one would have a separate source for each branch, but the source distribution function is unique, and it is further constrained by the GKL. The generalization of the single branch result follows directly from the methods of the calculus of variations with constraints, and is given by

$$s(z)=w(z)\sum_{k=1}^{3}\alpha_k\Psi_k^*(z)\,. \tag{19}$$

We then choose the complex constants α_k to maximize E_k/A_k, constrained by the conditions of the GKL and by the normalization condition of Eq. (18). The problem is thus reduced to finding the α_k which determine the source distribution function.

We may choose any one α_k to be real and determined by Eq. (18) with $s(z)$ from Eq. (19). Then we must add the GKL constraints which make the extremal problem nontrivial. The solution method will be detailed in a subsequent paper, but the process leads to unique values for the α_k and P_0 and hence to a unique source distribution function. We also note that the expression of Eq. (19) has the clear physical interpretation of being the *local* Kirchhoff's law for a nonuniformly magnetized plasma.

In Figs. 1 and 2, we show the quantity $|s(x)|^2$ for each case of Table I and Table II, where x is in centimeters and the origin is at the harmonic. The source strength increases in magnitude and width as the absorption increases with k_z for the ions and with T_e for the electrons. For the ion source distribution function, the nominally Gaussian shape is shifted due to mode conversion effects, while the shift in the electron synchrotron case is ap-

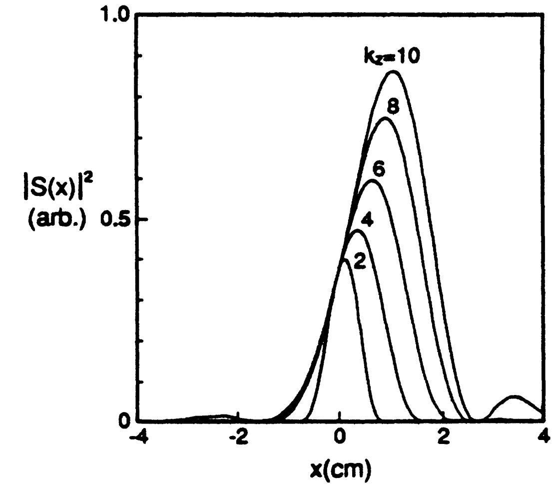

FIG. 1. Ion distributed source strengths $|s(x)|^2$ for values of k_z given in Table I.

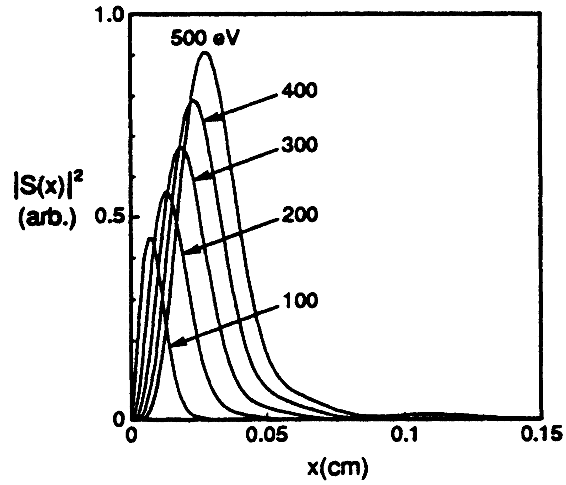

FIG. 2. Electron distributed source strengths $|s(x)|^2$ for values of T_e given in Table II.

parently due more to the relativistic shift and broadening than to mode conversion effects.

In conclusion, it has been shown that the effects of inhomogeneity in a mode conversion layer on emission cause it to differ substantially from the classical value in many cases. The reduction of the emission due to mode conversion effects is at least partially compensated due to the conversion of an incident Bernstein wave adding to the direct emission, but there is still asymmetry between the emission from the high- and low-field sides. For all cases, except in the limit of strong absorption, the concept of opacity must be revised to include the mode conversion effects. In the strong absorption case, the emission on each branch is blackbody, and this is common for the second harmonic where most ECE diagnostics operate, so no discrepancy is expected. Some ECE diagnostics are done at $\omega = 3\omega_{ce}$, however [3], where the absorption is not so strong and mode conversion effects are important. For ions, the mode conversion changes are nearly always important, since ion cyclotron harmonic emission is rarely blackbody and the compensating effects from the incident Bernstein wave are much weaker.

For overall synchrotron emission from a plasma, there are contributions from many harmonics, and typically all but the second will be reduced due to the generalized Kirchhoff's law. We find that the Bernstein wave, which emits more strongly than the fast wave branches, carries a nontrivial amount of energy, and leads to anomalous energy transport. Through the variational analysis developed and the discovery of the local Kirchhoff's law, we have also calculated for the first time the local source distribution function in a mode conversion layer.

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