Renormalization Theory for Eddy Diffusivity in Turbulent Transport

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We examine the derivation of eddy-diffusivity equations for transport of passive scalars in a turbulent velocity field. Our main contention is that, in the long-time–large-distance limit, the eddy-diffusivity equations can take very different forms according to the statistical properties of the subgrid velocity, and that these equations depend very sensitively on the interplay between spatial and temporal velocity fluctuations. Such crossovers can be represented in a "phase diagram" involving two relevant statistical parameters. Strikingly, the Kolmogorov-Obukhov statistical theory is shown to lie on a phase-transition boundary.

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Turbulent transport is ubiquitous in many processes in nature, ranging from atmospheric diffusion of clouds and the spreading of pollutants in large bodies of water [1] to ionic diffusion in high-energy plasmas like the Sun [2]. A fundamental feature in the physics of turbulent transport is the presence of a wide, self-similar spectrum of scales of motion (e.g., the atmospheric turbulence with a cascade of approximately 10^{11} active wave numbers). Predicting the macroscopic behavior of advected passive scalars and deriving "eddy-diffusivity" equations for large-scale motions are issues of theoretical and practical interest. In fact, passive turbulent transport is a reasonably simple situation where various field-theoretical methods of turbulence theory can be put to test [3–5]. On the practical side, the introduction of eddy diffusivities to model enhanced diffusion by inertial-range modes is necessary in turbulence calculations even with current supercomputers [6]. In this Letter, we show that the correct eddy-diffusivity equations for a given random velocity are determined by the joint values of two statistical parameters, ϵ and z, according to a suitable "phase diagram." This is due to the effect of the interplay of spatial and temporal velocity fluctuations at the subgrid level on the macroscopic scale.

A convenient starting point is to use the Kolmogorov-Obukhov [7,8] statistical theory of turbulence, which postulates the scaling relations

$$E(k)\propto k^{-5/3},\quad \omega(k)\propto k^{2/3} \tag{1}$$

for the kinetic energy $E(k)$ and the characteristic frequency $\omega(k)$ for velocity modes with wave number k in the inertial range $k_i<k<k_d$. Here k_i and k_d are, respectively, the reciprocals of the integral and the dissipation length scales. The ratio $\delta=k_i/k_d\ll 1$ is proportional to $\mathrm{Re}^{-3/4}$, where Re is the Reynolds number. Within a statistical framework, and using units of dissipation length and time scales, the turbulent velocity field $\mathbf{u}(\mathbf{x},t)$ can be written as

$$\mathbf{u}(\mathbf{x},t)=\int\int e^{-2\pi i(\mathbf{k}\cdot\mathbf{x}+\omega t)}d\hat{\mathbf{u}}(\mathbf{k},\omega)\,,$$

where $d\hat{\mathbf{u}}(\mathbf{k},\omega)$ is the random spectral measure [9] satisfying $d\hat{\mathbf{u}}(\mathbf{k},\omega)=d\hat{\mathbf{u}}(-\mathbf{k},-\omega)$, $\langle d\hat{\mathbf{u}}(\mathbf{k},\omega)\rangle=0$, and

$$\langle d\hat{u}_i(\mathbf{k},\omega)d\hat{u}_j(\mathbf{k}',\omega')\rangle =\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')[\delta_{ij}-k_ik_j/k^2]\frac{E(k,\omega)d\mathbf{k}\,d\omega}{(d-1)k^{d-1}\Omega_d}\,. \tag{2}$$

Here brackets denote statistical averaging, $k=|\mathbf{k}|$, d is the space dimensionality, Ω_d denotes the area of the d-dimensional sphere, and $E(k,\omega)$ is the energy density given by

$$E(k,\omega)=\begin{cases}\bar{U}^2k^{1-\bar{\epsilon}}[k^{-z}\Phi(\omega/k^z)], & \text{for } \delta\le k<1\,,\\ 0, & \text{if } k<\delta \text{ or } k>1\,,\end{cases} \tag{3}$$

where $\bar{\epsilon}=\frac{8}{3}$ and $z=\frac{2}{3}$, in accordance with (1). The function Φ is a positive structure function; its specific form is irrelevant for the present discussion. We assume, naturally, that $\Phi(0)>0$ and that $\int_{-\infty}^{+\infty}\Phi(\omega)d\omega<\infty$. Equation (3) is equivalent to stating that the autocorrelation function *in time* of the velocity modes at wave number k is

$$\langle d\hat{u}_i(\mathbf{k},t)d\hat{u}_j(\mathbf{k},t')\rangle=\bar{U}^2k^{1-\bar{\epsilon}}[\delta_{ij}-k_ik_j/k^2]\times f(k^z|t-t'|)d\mathbf{k}/(d-1)k^{d-1}\Omega_d\,,$$

where $f(\tau)$ (the Fourier transform of Φ) decays as $\tau\to\infty$. Thus, (2) and (3) simply express the fact that a cascade exists in wave-number space and that moreover the decorrelation in time of different modes is wavelength dependent with large-scale structures decorrelating more slowly than small ones, according to the value of z. Note that, from (3), the overall kinetic energy at wave number k is

$$E(k)=\int_{-\infty}^{+\infty}E(k,\omega)d\omega=\bar{U}^2\left(\int_{-\infty}^{+\infty}\Phi(\omega)d\omega\right)k^{1-\bar{\epsilon}},\quad \text{for } \delta<k<1\,,$$

and this is consistent with (1). This nondimensionalization by the dissipation scales in (3) leads to infrared divergences in the velocity spectrum as $\mathrm{Re}\to\infty$, or equivalently, $\delta \to 0$ which must be "renormalized" in order to obtain the eddy-diffusivity equations. The equation describing the evolution of a passive scalar $T(\mathbf{x},t)$ is

$$\frac{\partial T(\mathbf{x},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\nabla T(\mathbf{x},t) = D\Delta T(\mathbf{x},t),$$

where D denotes molecular diffusivity and $\Delta = \sum_i (\partial/\partial x_i)^2$. We assume that the initial data $T(\mathbf{x},t=0)$ varies slowly with respect to the integral length scale, *as is natural for a theory for eddy diffusivity*, so that $T(\mathbf{x},t=0)=T_0(\delta\mathbf{x})$ for some initial profile $T_0(\mathbf{x})$. We seek to determine a time-scaling function $\rho=\rho(\delta)$ such that the limit

$$\lim_{\delta\downarrow 0}\langle T(\delta^{-1}\mathbf{x},\rho^{-2}t)\rangle = \bar{T}(\mathbf{x},t) \tag{4}$$

is a function evolving according to an effective equation of motion, which describes the long-time–large-scale transport of the averaged scalar. [In particular, we should have $\bar{T}(\mathbf{x},t)\neq 0$ and $\bar{T}(\mathbf{x},t)\neq T_0(\mathbf{x})$ in (4) for $t>0$.] We refer to this problem as the *renormalization problem*, not to be confused with the renormalization-group (RNG) procedure [4] sometimes used to solve it.

It is useful to consider the exponents $\bar{\epsilon}$ and z as parameters that can vary in a neighborhood of the Kolmogorov values $\bar{\epsilon}=\frac{8}{3}$, $z=\frac{2}{3}$. In previous work [10,11], we introduced and rigorously analyzed the renormalization problem for *anisotropic "stratified" flows* with turbulent velocity statistics of the form $\mathbf{u}(\mathbf{x},t)=[u(x_2,t),0]$, with $\mathbf{x}=(x_1,x_2)$. We calculated the function $\rho=\rho(\delta)$ as well as the effective equations of motion for $\bar{T}(\mathbf{x},t)$ for $\bar{\epsilon}$ and z varying in the ranges $-\infty<\bar{\epsilon}<4$ and $z>0$. The results can be summarized in a "phase diagram" in the $(\bar{\epsilon},z)$ plane, in which the exponent $\nu=\nu(\bar{\epsilon},z)=\lim_{\delta\downarrow 0}(\log\rho/\log\delta)$ plays the role of an order parameter, in analogy with statistical mechanics. The diagram has several "phases," corresponding to different functional forms for $\nu(\bar{\epsilon},z)$ and different effective equations [10,11]. The portion of this phase diagram for $z<1$ is reproduced in Fig. 1. Region I corresponds to normal (Fickian) diffusion, $\rho(\delta)=\delta$. Here we focus attention on regions II and III and on the segment separating them, which contains the Kolmogorov value $(\frac{8}{3},\frac{2}{3})$. Rigorous analysis of the model [10,11] [which corresponds to an anisotropic version of (2) and (3), with $d=1$] shows the following: (i) In *region II* we have $\nu(\bar{\epsilon},z)=(4-\bar{\epsilon}-z)/2$ and the effective evolution equation is $(\partial/\partial t)\bar{T}=D^*_{\rm II}(\partial^2/\partial x_1^2)\bar{T}$, with an eddy diffusivity given by $D^*_{\rm II}=\Phi(0)\bar{U}^2/(2-\bar{\epsilon}-z)$; (ii) in *region III* we have $\nu(\bar{\epsilon},z)=1-\bar{\epsilon}/4$ and an effective diffusion equation $(\partial/\partial t)\bar{T}=D^*_{\rm III}(t)(\partial^2/\partial x_1^2)\bar{T}$, with $D^*_{\rm III}(t)=t\bar{U}^2[\int_{-\infty}^{+\infty}\Phi(\omega)d\omega]/(\bar{\epsilon}-2)$; and (iii) on the *boundary between regions II and III*, corresponding to $\bar{\epsilon}+2z=4$, $0<z<1$, the scaling function corresponds to $\nu(\bar{\epsilon},z)=(4-\bar{\epsilon}-z)/2=1-\bar{\epsilon}/4$, but the eddy diffusivity is given by

$$D^*_b(t)=\bar{U}^2\int_1^{+\infty}k^{1-\bar{\epsilon}-z}\left[\int_0^{k^z t}f(s)ds\right]dk\,.$$

The fact that the Kolmogorov regime corresponds to a phase boundary is not an artifice of the model discussed in Refs. [10,11], but rather a general property of homogeneous isotropic turbulent transport in arbitrary space dimensions. Indeed, the purpose of this Letter is to establish that *the renormalization problem for turbulent transport by isotropic, homogeneous random fields in d-space dimensions has exactly the same phase diagram as in Fig. 1* for $(\bar{\epsilon},z)$ in a vicinity of the Kolmogorov values. Moreover, there is a clear physical interpretation of regions II and III; region II corresponds to the statistical regime in which time-decorrelation effects are dominant and the Kubo formula [12] for the eddy diffusivity is exact after a suitable renormalization to remove the infrared divergence. On the other hand, region III corresponds to the regime for which time-decorrelation effects are completely negligible in the long-time–large-distance limit and hence G. I. Taylor's hypothesis of "frozen" turbulence [13] is rigorously valid. This result provides, in our opinion, a useful underpinning for understanding the dynamical implications of intermittency corrections to scaling [6–8,14] for the classical $k^{-5/3}$ law. Next, we describe the particulars of the renormalization problem for isotropic turbulence in regions II and III and on the segment separating them.

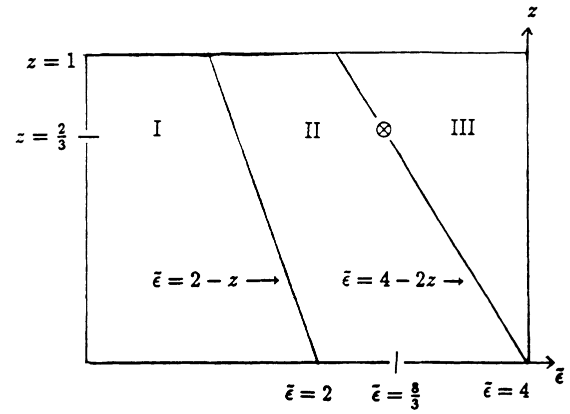

FIG. 1. The phase diagram for turbulent transport, for $-\infty<\bar{\epsilon}<4$, $0<z<1$. Region I corresponds to Fickian scaling $\nu(\bar{\epsilon},z)=1$. In region II, we have $\nu=(4-\bar{\epsilon}-z)/2$ and in region III, $\nu=1-\bar{\epsilon}/4$. The Kolmogorov regime $\bar{\epsilon}=\frac{8}{3}$, $z=\frac{2}{3}$ is located on the boundary between the latter regions and is marked by $\otimes$.

Renormalization in region II.—This region is defined by the inequalities $\bar{\epsilon}+z>2$, $\bar{\epsilon}+2z<4$, $z>0$. Inspection of the (Taylor) diffusivity in a second-order perturbation calculation shows divergence as $\delta\to 0$ with the usual diffusive scaling $\rho(\delta)=\delta$ so that anomalous diffusion occurs and the appropriate time-scaling function must satisfy $\rho(\delta)\ll\delta$. Motivated by the model in Refs. [10,11], we set $\rho(\delta)=\delta^{(4-\epsilon-z)/2}$, and consider the equation of motion for the rescaled function $\tilde{T}_\delta(\mathbf{x},t)=T(\delta^{-1}\mathbf{x},\rho^{-2}t)$ which can be written in the form

$$(\partial/\partial t)\tilde{T}_\delta(\mathbf{x},t)+[\sigma(\delta)]^{-1}\mathbf{V}(\mathbf{x},[\sigma(\delta)]^{-2}t)\cdot\nabla\tilde{T}_\delta(\mathbf{x},t) = \delta^{\bar{\epsilon}+z-2}D\Delta\tilde{T}_\delta(\mathbf{x},t)\,, \tag{5}$$

with $\sigma(\delta)=\delta^{-z/2}\rho=\delta^{(4-\bar{\epsilon}-2z)/2}$ and

$$\mathbf{V}(\mathbf{x},t)=\delta^{(\bar{\epsilon}/2)-1}\mathbf{u}(\delta^{-1}\mathbf{x},\rho^{-2}t). \quad (6)$$

Note that both $\sigma(\delta)$ and the rescaled diffusion coefficient tend to zero as $\delta\to 0$ precisely in region II. From (6), we calculate that the renormalized Kubo diffusivity D_{11}^* for (5) satisfies

$$\begin{aligned} dD_{11}^* &= \int_0^{+\infty}\langle \mathbf{V}(\mathbf{x},t)\cdot\mathbf{V}(0,0)\rangle dt \\ &= \delta^{\bar{\epsilon}+z-2}\int_0^{\infty}\langle \mathbf{u}(0,t)\cdot\mathbf{u}(0,0)\rangle dt \\ &= \delta^{\bar{\epsilon}+2-2}\bar{U}^2\Phi(0)\int_\delta^1 k^{1-\bar{\epsilon}-z}dk \\ &\cong \frac{\bar{U}^2\Phi(0)}{2-\bar{\epsilon}-z}. \end{aligned} \quad (7)$$

Thus, *after renormalization according to the anomalous scaling $\rho(\delta)=\delta^{(4-\bar{\epsilon}-z)/2}$, the velocity field in (6) satisfies the usual hypothesis for the validity of the Kubo-type weak coupling approximation. Namely, the temporal fluctuations are on a scale which is much faster than the spatial fluctuations, since $\sigma(\delta)\to 0$ for $\delta\to 0$, and the diffusivity in (7) is finite.* We conclude that, as $\delta\to 0$, we have $\lim_{\delta\downarrow 0}\langle\tilde{T}_\delta(\mathbf{x},t)\rangle=\bar{T}(\mathbf{x},t)$, where this function satisfies the effective diffusion equation $(\partial/\partial t)\times\bar{T}(\mathbf{x},t)=D_{11}^*\Delta\bar{T}(\mathbf{x},t)$ with the eddy diffusivity

$$D_{11}^*=\bar{U}^2\Phi(0)/d(\bar{\epsilon}+z-2). \quad (8)$$

Analogous results for equations such as (5) have been obtained mathematically by several authors [15].

Renormalization in region III.—In the adjacent region, defined by the inequalities $4-2z<\bar{\epsilon}<4$, the appropriate scaling function suggested by the model is $\rho(\delta)=\delta^{1-\bar{\epsilon}/4}$. To justify the scaling for multidimensional isotropic turbulence, we note that the scalar $\tilde{T}_\delta(\mathbf{x},t)$ satisfies the transport equation

$$(\partial/\partial t)\tilde{T}_\delta(\mathbf{x},t)+\mathbf{W}^{(\delta)}(\mathbf{x},t)\cdot\nabla\tilde{T}_\delta(\mathbf{x},t)=\delta^{\epsilon/2}D\Delta\tilde{T}_\delta(\mathbf{x},t), \quad (9)$$

where $\mathbf{W}^{(\delta)}(\mathbf{x},t)$ is the auxiliary velocity field given by

$$\mathbf{W}^{(\delta)}(\mathbf{x},t)=\delta^{(2-\epsilon)/2}\mathbf{u}(\delta^{-1}\mathbf{x},\rho^{-2}t).$$

Because of the inequalities satisfied by $\bar{\epsilon}$ and z in region III, the renormalized fields $\mathbf{W}^{(\delta)}$ enjoy the following properties: (i) the mean-square velocity $\langle|\mathbf{W}(\mathbf{x},t)|^2\rangle$ remains uniformly bounded as $\delta\to 0$; and (ii) under reasonable statistical mixing assumptions [16] on the original velocity $\mathbf{u}(\mathbf{x},t)$, it can be shown [17] that the random fields $\mathbf{W}_\delta(\mathbf{x},t)$ converge in distribution to a time-independent Gaussian field $\mathbf{W}^{(0)}(\mathbf{x})$ with covariance

$$\begin{aligned} R_{ij}^{(0)}(\mathbf{y}) &= \langle W_i^{(0)}(\mathbf{x}+\mathbf{y})W_j^{(0)}(\mathbf{x})\rangle \\ &= c\bar{U}^2\int_{k>1}k^{2-d-\epsilon}(\delta_{ij}-k_ik_j/k^2) \\ &\quad\times e^{2\pi i\mathbf{k}\cdot\mathbf{y}}\frac{d\mathbf{k}}{(d-1)\Omega_d}, \end{aligned} \quad (10)$$

where $c=\int_{-\infty}^{+\infty}\Phi(\omega)d\omega$. These properties are immediate consequences of the fact that $\mathbf{W}^{(\delta)}(\mathbf{x},t)$ has an $O(1)$ covariance $R_{ij}^{(\delta)}(\mathbf{y},\tau)$ given by

$$\begin{aligned} R_{ij}^{(\delta)}(\mathbf{y},\tau) &= \langle W_i^{(\delta)}(\mathbf{x}+\mathbf{y},t+\tau)W_j^{(\delta)}(\mathbf{x},t)\rangle \\ &= \bar{U}^2\int_{1<k<\delta^{-1}}\int k^{2-d-\bar{\epsilon}}\left\{(k)^{-z}\Phi\left[\frac{\omega}{(k)^z}\right]\right\}P_{ij}(\mathbf{k})e^{-2\pi i(\mathbf{k}\cdot\mathbf{y}+\delta^\theta\omega\tau)}\frac{d\mathbf{k}\,d\omega}{(d-1)\Omega_d}, \end{aligned} \quad (11)$$

where $\theta=(\epsilon+2z-4)/2$, $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2$. From the last equality in (11), $R_{ij}^{(\delta)}(\mathbf{y},\tau)$ converges to $R_{ij}^{(0)}(\mathbf{y})$ as the cutoff $\delta\to 0$, and the convergence in distribution of $\mathbf{W}^{(\delta)}$ to $\mathbf{W}$ follows. From (9), molecular diffusion is negligible in the limit $\delta\downarrow 0$ so we anticipate from (10) and (11) that the limiting eddy-diffusivity equation is transport by the steady incompressible velocity field $\mathbf{W}$; thus Taylor's hypothesis of frozen turbulence is valid for region III. Furthermore, *given a nonconstant initial datum $T_0(\mathbf{x})$, for any sequence $\delta_n\searrow 0$ such that the limit $\lim_{\delta_n\downarrow 0}\langle\tilde{T}_{\delta_n}(\mathbf{x},t)\rangle=\bar{T}(\mathbf{x},t)$ exists, we have $\bar{T}(\mathbf{x},t)\neq 0$ and* $\bar{T}(\mathbf{x},t)\neq T_0(x)$ and, therefore, the anomalous exponent $v(\bar{\epsilon},z)=1-\epsilon/4$ is indeed correct for multidimensional transport in region III. These results can be established with full mathematical rigor [17] but are subtle because the limiting steady velocity field $\mathbf{W}(x)$ is not smooth. In general, the eddy-diffusivity equation is nonlocal in region III. Rigorous upper bounds and a characterization for the limiting Green's function utilizing Stieltjes measure formulas [18,19] under additional assumptions are also presented in Ref. [17].

The boundary between II and III.—For $(\bar{\epsilon},z)$ on the interface between regions II and III, i.e., $\bar{\epsilon}+2z=4$, $0<z<1$, we take $\rho(\delta)=\delta^{(4-\epsilon-z)/2}=\delta^{1-\epsilon/4}$ (the exponent is continuous). The auxiliary velocity fields $\mathbf{W}^{(\delta)}(\mathbf{x},t)$ in (10) have the covariance $R_{ij}^{(\delta)}(\mathbf{y},\tau)$ in (11) with $\theta=0$. Therefore, $\{\mathbf{W}^{(\delta)}(\mathbf{x},t)\}$ converges [17] as $\delta\to 0$ to a Gaussian field $\mathbf{W}^{(0)}(\mathbf{x},t)$, under suitable mixing assumptions. Moreover, $\mathbf{W}^{(0)}(\mathbf{x},t)$ has covariance

$$R_{ij}^{(0)}(\mathbf{y},\tau)=\bar{U}^2\int_{k>1}\int k^{2-d-\epsilon-z}\Phi\left[\frac{\omega}{k^z}\right]P_{ij}(\mathbf{k})e^{2\pi i(\mathbf{k}\cdot\mathbf{y}+\omega\tau)}d\mathbf{k}\,d\omega.$$

In particular, the limiting random velocity field is time dependent. Following the reasoning described earlier for region

III, it can be established rigorously [17] that the eddy-diffusivity equation corresponds to transport by this time-dependent random velocity field and also that the correct anomalous exponent is $\nu(\tilde{\epsilon},z)=(4-\tilde{\epsilon}-z)/2=1-\tilde{\epsilon}/4$. In particular, the *temporal fluctuations of the turbulent velocity* $\mathbf{u}(\mathbf{x},t)$ *in Kolmogorov-Obukhov turbulence* ($\tilde{\epsilon}=\frac{8}{3}$, $z=\frac{2}{3}$) *are not irrelevant in the long-time–large-distance limit.* Hence, the boundary between regions II and III ($\tilde{\epsilon}+2z=4$, $0<z<1$) has remarkable crossover properties [10,11,20]. Physically, it corresponds to statistical regimes for turbulent velocities in which the characteristic wave-number-dependent frequency $\omega(k)=k^z$ satisfies the dimensional relation

$$\omega(k)\propto k\left[\int_k^{+\infty}E(k')dk'\right]^{1/2},$$

where $E(k)\propto k^{1-\tilde{\epsilon}}$, $2<\tilde{\epsilon}<4$.

In conclusion, we have shown that the large-scale-long-time properties of passive scalars in turbulent transport depend crucially on the form of the self-similar energy spectrum $E(k,\omega)$ through the exponents $\tilde{\epsilon}$ and z, which characterize, respectively, the energy spectrum and the turnover frequency of modes of different wavelengths. Eddy-diffusivity equations for isotropic, homogeneous turbulence with $(\tilde{\epsilon},z)$ in regions II and III have been established, with scaling exponents $\nu(\tilde{\epsilon},z)$ which agree with the ones for the simple stratified model [10,11] in those regions. The relations $\nu=1-\tilde{\epsilon}/4$, $\max(2,4-2z)<\tilde{\epsilon}<4$, $\nu=(4-\tilde{\epsilon}-2)/2$, and $2-z<\tilde{\epsilon}<4-2z$ imply that the knowledge of two numbers among $\tilde{\epsilon},z,\nu$ determines the third one. Thus, we find that the statistical regimes which are consistent with the Richardson $X^2\sim T^3$ law [21]—i.e., $\nu=\frac{1}{3}$—must necessarily satisfy either $\tilde{\epsilon}=\frac{8}{3}$, $z\geq\frac{2}{3}$ or $\tilde{\epsilon}+z=\frac{10}{3}$, $0\leq z\leq\frac{2}{3}$. This result agrees with the corresponding theory for relative diffusion of pairs of particles which was established in the model of stratified flows for all values of $(\tilde{\epsilon},z)$ [11]. It can also be shown that, *for* $z<1$, and thus in a neighborhood of the Kolmogorov-Obukhov regime, the theory presented here is Galilean invariant, i.e., it is not modified if a uniform mean flow $\bar{\mathbf{u}}$ is added to the turbulent velocity.

The rigorous renormalization theory developed here provides an important nontrivial test problem for the capability of RNG and renormalized perturbation theory methods to reproduce features of the eddy diffusivity [20].

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