# Self-Trapping of Traveling-Wave Pulses in Binary Mixture Convection

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Localized traveling-wave trains (LTW) have been observed in various experiments on binary mixture convection. I show that the commonly used complex Ginzburg-Landau equations, which fail to describe a characteristic feature of LTW—their extremely slow drift—break down in these systems and I derive a new set of coupled equations which takes into account the slow dynamics of the concentration field. It possesses slow LTW over a wide range of parameters. In addition, it supports LTW even if it has only real coefficients and is therefore far from the nonlinear Schrödinger limit.

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Convection in binary mixtures has turned out to be a very rich pattern-forming system, in particular in the regime where convection arises through a Hopf bifurcation. In most experimental situations this bifurcation is subcritical and leads to traveling waves. The most striking phenomenon in these systems is the appearance of coherent structures in the form of localized traveling-wave trains (LTW). They have been first observed in finite geometries [1] where their origin has been attributed to the convective nature of the instability and the reflection of waves at the side walls [2].

It came as a great surprise when it was found that even in annular geometries localized wave trains can arise [3]. Meanwhile, such LTW have been found by various experimental groups [4] as well as in full numerical simulations of the two-dimensional Navier-Stokes equations [5] over a whole range of values of the separation ratio $S$ and of the Rayleigh number $R$ [6]. More precisely, two classes of LTW seem to exist: discrete sets of pulselike structures and a continuum of LTW of seemingly arbitrary length which resemble a pair of fronts connecting the convective and the conductive states. All of these structures share one feature: Their drift velocity is extremely small, if not zero altogether [7].

Theoretically, one would like to understand the structures using simpler model equations like the Ginzburg-Landau equations which can be derived from the basic Navier-Stokes equations if the convection amplitude is small. In the present system a quantitative description may only be expected in the (small) parameter regime in which the subcriticality of the bifurcation is weak. Nevertheless, even a semiquantitative or at least qualitative understanding would be of great interest. In equilibrium systems, for which the coefficients of the corresponding Ginzburg-Landau equation are real, localized structures exist only as "critical droplets" which are unstable. Therefore, great progress was achieved when it was realized by Thual and Fauve that in the complex Ginzburg-Landau equation (CGL) for the convective amplitude $A$,

$$\partial_t A + s\partial_x A = d\partial_x^2 A + aA + cA|A|^2 + pA|A|^4 + d_2|A|^2\partial_x A + d_3 A^2 \partial_x A^* , \qquad (1)$$

which is an extension of the nonlinear Schrödinger equation, pulses can be stable due to dispersion and nonlinear frequency renormalization [8,9]. Their result triggered very detailed analyses of the CGL focusing on pulses as well as front pairs [10–12]. In all of these analyses a central problem, however, remains: Any localized solution of the CGL drifts with a velocity which is essentially determined by the linear group velocity $s$ of the waves. To state it more precisely, the drift velocity $v$ is to lowest order given by $s$ and is at higher order affected by the nonlinear gradient terms [13],

$$v = s + f(d_1, d_2) . \qquad (2)$$

The linear group velocity, however, is much larger [14,15] (up to a factor of 30) than the observed drift velocity of the LTW. Therefore, within the CGL a delicate balance of $s$ and $f(d_1,d_2)$ would have to occur *over a range of S and R*. In addition, one should also be able to find *fast* LTW in some range, which has not been reported. This indicates that the CGL may not be adequate to describe LTW in these binary mixtures even on a qualitative level [7,16].

In this Letter I suggest that the slow drift velocity of the LTW is due to an additional slow time scale in the binary mixture systems which leads to an additional dynamical degree of freedom beyond the convective amplitude $A$. It corresponds to a concentration mode and is due to the small Lewis number $L$ which measures the ratio of molecular to thermal diffusion. The relevance of the concentration field was already noted in the full numerical simulations of the Navier-Stokes equations by Barten, Lücke, and Kamps [5]. Considering the limit of small Lewis number, I derive a new set of *coupled* equations for the convective amplitude $A$ and the concentration mode $C$. Numerical simulations of these equations show that for small $L$ the pulse velocity is drastically reduced. What is more, the effect of the group velocity $s$ on the pulse velocity is strongly reduced as compared to the conventional CGL.

Expanded around the conductive state, the Navier-Stokes equations together with the equation for the temperature deviation $\theta$ and the concentration field $c = \eta + \theta$

read in the simplified case of infinite Prandtl number [15]

$$\begin{pmatrix} -\Delta^2 & -(1+S)\partial_x & -S\partial_x \\ -R\partial_x & \partial_t-\Delta & 0 \\ -R\partial_x & \partial_t & \partial_t - L\Delta \end{pmatrix}\begin{pmatrix}\phi\\ \theta\\ \eta\end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{N}\theta \\ \mathcal{N}(\eta+\theta)\end{pmatrix}, \quad (3)$$

where $\mathcal{N}=\partial_z\phi\partial_x-\partial_x\phi\partial_z$ and $\phi$ denotes the stream function. For sufficiently negative separation ratio the linear stability analysis of the conductive state yields a Hopf bifurcation. For the following it is crucial to notice that for vanishing molecular diffusion ($L=0$) additional critical modes arise: Any mode $(0,0,g(z))^T$ which has no convective or thermal contribution and which depends only on the vertical coordinate $z$ has a zero eigenvalue. Of course, these eigenvalues cannot become positive and the corresponding modes never destabilize the basic conductive state. Nevertheless, their dynamics is slow and they cannot be eliminated adiabatically in the limit $L\to 0$ at fixed $\epsilon$ with $R=R_c+\epsilon^2R_2$. Thus, they constitute additional dynamical degrees of freedom of the system. To derive the corresponding amplitude equations one has to take into account that for nonzero $L$ all of the concentration modes decay and one need only keep those which are driven by the convective amplitude $A$.

To minimize the technical effort I consider here the case of free-slip-permeable boundary conditions. This allows a simple analytical treatment of the problem which shows the structure of the problem. No particular significance will be attached to the specific values obtained for the coefficients in this way. The fields $\phi$, $\theta$, and $\eta$ are expanded as

$$\begin{pmatrix}\phi\\ \theta\\ \eta\end{pmatrix} = \epsilon\sin\pi z\, e^{iqx}\left[Ae^{-i\omega t}\begin{pmatrix}1\\ \psi_1^{(r)}\\ \psi_2^{(r)}\end{pmatrix} + Be^{i\omega t}\begin{pmatrix}1\\ \psi_1^{(l)}\\ \psi_2^{(l)}\end{pmatrix}\right] + \epsilon C\sin 2\pi z\begin{pmatrix}0\\0\\1\end{pmatrix} + \text{higher-order terms}, \quad (4)$$

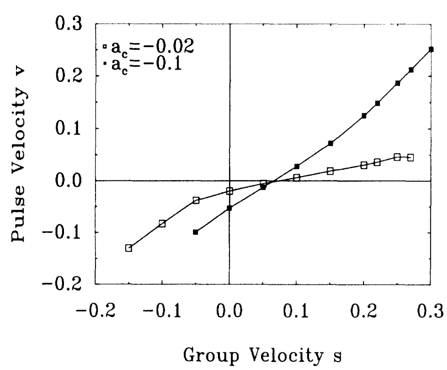


FIG. 1. Drift velocity $v$ vs group velocity $s$ for a pulse in the complex Ginzburg-Landau equation with coefficients $d=0.15+i$, $a=-0.24$, $f=0.5$, $c=2.4+2i$, $p=-1.65+2i$, $d_c=0.1$, and $h_2=0.3$.

where $A$ and $B$ are the convective amplitudes of right- and left-traveling waves, respectively. The eigenfunction of the additional concentration mode $C$ is determined by the nonlinear coupling term $(\partial_z\phi\partial_x-\partial_x\phi\partial_z)\eta$ which drives such a mode. A detailed derivation (using the scaling $L=\epsilon^2L_2$) will be given elsewhere [17]. The form of the equations can be derived on symmetry grounds. Translation symmetry in space and time and reflection symmetry in space require that the resulting equations be invariant under the transformations $T_{\Delta x}(A,B,C)=(e^{iq\Delta x}A,e^{iq\Delta x}B,C)$, $T_{\Delta t}(A,B,C)=(e^{-i\omega\Delta t}A,e^{i\omega\Delta t}B,C)$, and $\kappa(A,B,C)=(B^*,A^*,C)$, which leads up to cubic order to

$$\partial_tA+s\partial_xA=d\partial_x^2A+(a+fC)A+cA|A|^2+gA|B|^2, \quad (5)$$

$$\partial_tC=d_c\partial_x^2C+a_cC+h_2\partial_x(|A|^2-|B|^2) + h_4(A\partial_xA^*-B^*\partial_xB)+(h_1+h_3C)(|A|^2+|B|^2), \quad (6)$$

and a similar equation for $B$. The detailed analysis shows that for free-slip-permeable boundary conditions $h_1=0=h_4$ and $a_c=-4\pi^2L_2$. The fact that $a_c$ is proportional to $L$ is a general result independent of the boundary conditions which reflects the linear damping of the concentration mode by molecular diffusion. It is of interest to note that its time scale is given by the *vertical* diffusion time and not a diffusion time related to the length of the LTW. This is due to the fact that the concentration mode which is driven by the convective field has zero vertical average. Thus, for the case of idealized boundary conditions and infinite Prandtl number discussed here, it is not the global mean flow [18] which modifies the dynamics. In the general case additional contributions may arise.

To show the effect of the additional field $C$, I present numerical simulations for the case $B=0$, i.e., for a pure right-traveling wave which is coupled to the concentration mode. The lateral boundary conditions are taken periodic and for simplicity I take $h_1=h_3=h_4=0$. The general case is deferred to later work. I discuss two cases. First the coefficients are chosen such that the CGL by itself supports stable pulses [9]. Then the coefficients are chosen real, i.e., the Ginzburg-Landau equation for $A$ alone would not possess any stable localized solutions.

Figure 1 presents the main result. With the coefficients of the equation for $A$ chosen as in [9], i.e., the bifurcation is taken subcritical and the quintic term $p|A|^4A$ is retained, it gives the drift velocity $v$ of the pulse as a function of the group velocity $s$ for two values of the damping coefficient $a_c$ (which is proportional to the Lewis number). The remaining coefficients are chosen as $d_2=0$, $d_3=0$, $f=0.5$, $d_c=0.1$, and $h_2=0.3$. Note that for large damping the $C$ mode could be eliminated adiabatically, $C\propto\partial_x|A|^2/a_c$, and one would obtain the conventional CGL for $A$ alone with changed coefficients (in particular, nonlinear gradient terms are also generated). This case is represented by $a_c = -0.1$. As to be expected, the pulse velocity changes essentially by the same amount as the group velocity, i.e., $dv/ds \cong 1$ [cf. (2)] and correspondingly the pulse velocity is only small for a small range of parameters. If the Lewis number is reduced, however, this behavior changes drastically: For $a_c = -0.02$ the drift velocity is not only significantly reduced but it also *depends only weakly* on $s$ ($dv/ds \cong 0.3$). Such an effect can clearly not be achieved with a single CGL.

In Fig. 2 the pulse shapes and their concentration fields are compared for the two different Lewis numbers of Fig. 1 ($s=0.2$). Clearly the $C$ field is more pronounced for small Lewis number. Its shape makes the reduction in the pulse velocity rather intuitive. Because of the driving term $\partial_x |A|^2$ the $C$ field is reduced on one side of the pulse. This reduces the local growth rate and leads to a barrier for the pulse. Now, if this barrier is *ahead* of the pulse it becomes *trapped* by its own concentration field. This effect resembles the interpretation of the numerical simulations [5]. It is remarkable that the drift velocity can even be opposite to the group velocity ($0 \le s \le 0.07$). This may be related to the experimental finding that the drift velocity can change sign with decreasing Rayleigh number [16]. On the other hand, if the barrier is in back of the pulse its drift velocity is not reduced significantly ($s$ negative) and $dv/ds \cong 1$. It is striking that the shape of the convective amplitude $|A|$ is barely influenced by the $C$ field despite its strong effect on the velocity. Measuring solely the flow velocity would therefore give little indication of the relevance of the $C$ field in determining the velocity of the pulse. Thus, measurements of the local concentration become very important for analyzing this system [19].

An interesting effect is obtained when increasing the group velocity above $s=0.27$. The pulse envelope becomes deformed by the trapping and starts to oscillate at the trailing edge due to the occurrence of phase slips [17].

With real coefficients the Ginzburg-Landau equation (5) by itself cannot support any stable localized structures. However, if it is coupled to the $c$ mode this need not be the case any more. The profile of a stable LTW obtained in this way is shown in Fig. 3. The group velocity $s$ pushes the LTW to the right against the barrier formed by the $C$ mode ($C<0$ lowers the growth rate). As a result of the gradient in the growth rate the pulse actually drifts to the left. Again its velocity depends only weakly on $s$.

In conclusion, it was shown that the expansion leading to the commonly used CGL breaks down when the Lewis number becomes too small. This is due to a concentration mode of order $\epsilon^3 \partial_x |A|^2/L$ which diverges for $L \to 0$ at fixed $\epsilon$. If one keeps this mode as an independent dynamical variable no divergence occurs. The resulting coupled equations are very promising. For small Lewis number they possess generically *slow* pulse solutions, the velocity of which depends only weakly on the linear group velocity. In addition, the $C$ mode allows stable LTW even for real coefficients. Thus, their stabilization mechanism appears to be different from that of pulses or front pairs in the (perturbed) nonlinear Schrödinger equation [10-12]. It resembles that of localized drift waves arising from a secondary parity-breaking bifurcation. There the second field is the underlying wave number which also influences the local growth rate of the wave amplitude. In combination with the drift of the pattern this leads to localization [20].

Clearly, more work is necessary. In particular, analytical results for the pulse velocities and their shapes are desirable. It is expected that the concentration mode will have a strong influence on the interaction of pulses. In particular, bound states of left- and right-traveling pulses as observed in experiments [16] seem to be possible. In addition, the stability behavior of extended traveling waves will be changed by this mean field. Also, the coefficients should be calculated for realistic boundary conditions. These questions will be addressed in future work.

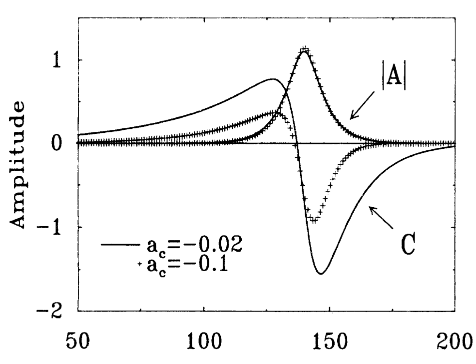


FIG. 2. Pulse shape for "large" and "small" Lewis number ($s=0.2$, other coefficients as in Fig. 1).

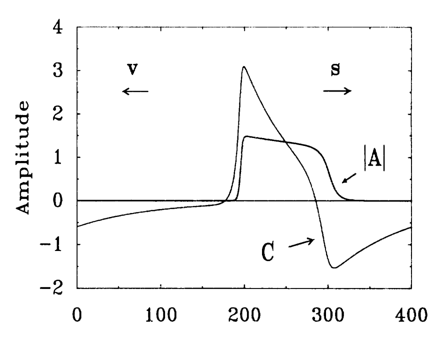


FIG. 3. Localized traveling wave for *real* Ginzburg-Landau equation with $C$ field. $s=0.8$, $d=0.15$, $a=-0.24$, $f=1.0$, $c=2.4$, $p=-1.65$, $d_c=0.1$, $a_c=-0.02$, and $h_2=0.3$.

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